

Lecture 1

1.1 Introduction

We will talk about n -variable calculus, that is, calculus on $\mathbb{R}^n$. Recall the following:

- The setting is,

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}}_{n \text{ times}} = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R} \forall i = 1, 2, \dots, n\}$$

- Analysis on $\mathbb{R}$ consisted of ideas like open sets, compact sets, convergence, limits, differentiability, integrability etc.
- $\mathbb{R}^n$ is an n -dimensional inner product space over $\mathbb{R}$, with the standard orthonormal basis $\{e_j\}_{j=1}^n$

Extending the analytic ideas to $\mathbb{R}^n$ exploiting the algebraic structure is the matter of this course, which further gives way to differential geometry.

1.2 Review: $\mathbb{R}^n$ as a vector space

- (i) The standard orthonormal basis of $\mathbb{R}^n$ is $\{e_i\}_{i=1}^n$.
- (ii) For all $x \in \mathbb{R}^n$, there is a unique representation

$$x = \sum_{i=1}^n x_i e_i, \quad x_i \in \mathbb{R}$$

Thus we identify x with the *coordinates* $(x_1, x_2, \dots, x_n)$.

- (iii) Euclidean inner product on $\mathbb{R}^n$:

For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we define

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i$$

1.3 Linear Functions

In doing analysis on $\mathbb{R}$, the main motive was to study functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and their properties, namely continuity, differentiability, integrability etc. We now wish to do the same for functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for arbitrary natural numbers n, m .

Two easy examples of such functions are:

(i) Constant maps

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\forall x \in \mathbb{R}^n, f(x) = a, a \in \mathbb{R}^m$$

(ii) Linear maps

A function $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if for all $\alpha \in \mathbb{R}, x, y \in \mathbb{R}^n$, $L(\alpha x + y) = \alpha L(x) + L(y)$.

It turns out that linear maps are useful in understanding most other ‘nice’ functions, and so we now look at these in more detail.

Let L be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Consider the domain $\{\alpha x + y \mid \alpha \in \mathbb{R}\}$, that is, the line through y in the direction of x . The image under L is,

$$\{\alpha Lx + Ly \mid \alpha \in \mathbb{R}\}$$

which is the line through Ly in the direction of Lx . Hence, L maps lines to lines.

Exercise. Is the converse also true?

Matrix representation of a linear map

Let $L : \mathbb{R} \rightarrow \mathbb{R}$ be a linear map. Then,

$$L(x) = xL(1) \quad \forall x \in \mathbb{R}$$

Therefore,

$$\mathcal{L}(\mathbb{R}, \mathbb{R}) := \{\text{set of all linear maps from } \mathbb{R} \text{ to } \mathbb{R}\} \leftrightarrow \mathbb{R}$$

Now consider the general case; let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. If we fix the bases $\{e_j\}_{j=1}^n$ of $\mathbb{R}^n$ and $\{e_i\}_{i=1}^m$ of $\mathbb{R}^m$, L is determined uniquely by the equations

$$Le_j = \sum_{i=1}^m a_{ij} e_i$$

and hence,

$$L \leftrightarrow (a_{ij})_{m \times n} \in M_{m,n}(\mathbb{R})$$

1.4 Analytic ideas in $\mathbb{R}^n$

We have the Euclidean norm on $\mathbb{R}^n$ defined by,

$$\|x\| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} \quad \forall x \in \mathbb{R}^n$$

This induces the metric given as,

$$d(x, y) = \|x - y\| = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}} \quad \forall x, y \in \mathbb{R}^n$$

Theorem 1.4.1 (Cauchy-Schwarz Inequality)

For all $x, y \in \mathbb{R}^n$,

$$\langle x, y \rangle \leq \|x\| \|y\|$$

Proof. Consider $x, y \in \mathbb{R}^n$. We have,

$$\sum_{i=1}^n \sum_{j=1}^n (x_i y_j - x_j y_i)^2 \geq 0$$

But the left-hand side is, after expanding,

$$\sum_{i,j=1}^n x_i^2 y_j^2 + \sum_{i,j=1}^n x_j^2 y_i^2 - 2 \sum_{i,j=1}^n x_i x_j y_i y_j = 2\|x\|^2 \|y\|^2 - 2\langle x, y \rangle^2$$

which gives the desired inequality. $\square$

Note: The proof shows that equality holds only if there is $\lambda \in \mathbb{R}$ such that for all i , either $x_i = \lambda y_i$ or $y_i = \lambda x_i$.

Recall the triangle inequality for $\mathbb{R}$, for all $x, y \in \mathbb{R}$

$$|x + y| \leq |x| + |y|$$

Theorem 1.4.2 (Triangle inequality for $\mathbb{R}^n$)

For all $x, y \in \mathbb{R}^n$,

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof. We have,

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \quad (\text{Cauchy Schwarz inequality}) \\ \implies \|x + y\| &\leq \|x\| + \|y\| \end{aligned}$$

which is the desired inequality. $\square$

The following is a technical result, which can be thought of as an analogue of the Lipschitz condition for linear maps on $\mathbb{R}^n$. It hints towards continuity of linear maps, and we will see that it is indeed so later.

Theorem 1.4.3

Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. There is $M > 0$ such that

$$\|Lx\| \leq M\|x\| \quad \forall x \in \mathbb{R}^n$$

Proof. We have, for $x = \sum_{i=1}^n x_i e_i$,

$$\begin{aligned} \|Lx\| &= \left\| \sum_{i=1}^n x_i L e_i \right\| \\ &\leq \sum_{i=1}^n |x_i| \|L e_i\| \quad (\text{Triangle inequality}) \\ \implies \|Lx\| &\leq \|x\| \left(\sum_{i=1}^n \|L e_i\|^2 \right)^{\frac{1}{2}} \quad (\text{Cauchy Schwarz inequality}) \end{aligned}$$

Taking $M = \left(\sum_{i=1}^n \|L e_i\|^2 \right)^{\frac{1}{2}}$, we get the result. $\square$