

Lecture 2

2.1 Distance and Topology in $\mathbb{R}^n$

Using the inner product on $\mathbb{R}^n$, we get the Euclidean distance

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

We wish to extend the notions of $(\mathbb{R}, |\cdot|)$ to $(\mathbb{R}^n, \|\cdot\|)$. We already know from Lecture 1 that the triangle inequality holds,

$$\|x + y\| \leq \|x\| + \|y\|$$

Definition 2.1.1 ► Open balls

The open ball centered at $a \in \mathbb{R}^n$ of radius r is,

$$B_r(a) = \{x \in \mathbb{R}^n \mid \|x - a\| < r\}$$

Exercise. Show that open balls are convex sets.

Definition 2.1.2 ► Open sets

A set $\mathcal{O} \subseteq \mathbb{R}^n$ is open if $\mathcal{O} = \emptyset$ or for all $x \in \mathcal{O}$, there is $r > 0$ such that $B_r(x) \subseteq \mathcal{O}$.

Example 2.1.1

- (i) Any open ball is open.
- (ii) We define open boxes in $\mathbb{R}^n$ to be the subsets of the form $\prod_{i=1}^n (a_i, b_i)$. Any open box is open.

Definition 2.1.3 ► Convergence of Sequences

Let $\{x_m\}_{m \in \mathbb{N}} \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say $x_m \rightarrow x$ if for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\begin{aligned} \|x_m - x\| &< \varepsilon \quad \forall m \geq N \\ \iff d(x_m, x) &< \varepsilon \quad \forall m \geq N \\ \iff x_m &\in B_\varepsilon(x) \quad \forall m \geq N \end{aligned}$$

Exercise. Show that the limit of a sequence in $\mathbb{R}^n$ is unique whenever it exists.

Definition 2.1.4 ► Limit points

We define the deleted ε -neighbourhood of $a \in \mathbb{R}^n$ to be $D_\varepsilon(a) = B_\varepsilon(a) \setminus \{a\}$. The point a is a limit point of $S \subseteq \mathbb{R}^n$ if for all $\varepsilon > 0$, $D_\varepsilon(a) \cap S \neq \emptyset$. If we do not delete a , we get isolated

points.

Definition 2.1.5 ► Projections

For all $i \in \{1, 2, \dots, n\}$ we define the maps

$$\begin{aligned}\Pi_i : \mathbb{R}^n &\rightarrow \mathbb{R} \\ x = (x_1, \dots, x_n) &\mapsto x_i\end{aligned}$$

Π_i is called the projection onto the i th coordinate.

Theorem 2.1.1

Let $\{x_m\}_{m \in \mathbb{N}} \cup \{x\} \subseteq \mathbb{R}^n$. Then,

$$\begin{aligned}x_m &\longrightarrow x \\ \iff \Pi_i(x_m) &\longrightarrow \Pi_i(x) \forall i \in \{1, 2, \dots, n\}\end{aligned}$$

Proof. Assume $x_m \longrightarrow x$. Now, for all $j \in \{1, 2, \dots, n\}$,

$$\begin{aligned}\|x_m - x\|^2 &= \sum_{i=1}^n |\Pi_i(x_m) - \Pi_i(x)|^2 \geq |\Pi_j(x_m) - \Pi_j(x)|^2 \\ \implies |\Pi_j(x_m) - \Pi_j(x)| &\longrightarrow 0 \\ \implies \Pi_j(x_m) &\longrightarrow \Pi_j(x)\end{aligned}$$

Now assume $\Pi_j(x_m) \longrightarrow \Pi_j(x)$ for all $j \in \{1, 2, \dots, n\}$. Then,

$$\begin{aligned}|\Pi_j(x_m) - \Pi_j(x)| &\longrightarrow 0, \forall j \in \{1, 2, \dots, n\} \\ \implies \sum_{i=1}^n |\Pi_i(x_m) - \Pi_i(x)|^2 &\longrightarrow 0 \\ \implies \|x_m - x\|^2 &\longrightarrow 0 \\ \implies x_m &\longrightarrow x\end{aligned}$$

□

Definition 2.1.6 ► Closed sets

A set $C \subseteq \mathbb{R}^n$ is closed if $\mathbb{R}^n \setminus C$ is open.

Exercise. Show that a set $C \subseteq \mathbb{R}^n$ is closed iff $\forall \{x_m\}_{m \in \mathbb{N}} \subseteq C$ with $x_m \longrightarrow x$ for some $x \in \mathbb{R}^n$, we have $x \in C$.

Exercise. Show that:

- (1) Arbitrary union of open sets is open.
- (2) Finite intersection of open sets is open.
- (3) Arbitrary intersection of closed sets is closed.
- (4) Finite union of closed sets is closed.
- (5) Any finite subset of $\mathbb{R}^n$ is closed.

Definition 2.1.7 ► Interior of a set

Let $\phi \neq S \subseteq \mathbb{R}^n$. The interior of S is,

$$\text{Int}(S) = \{a \in S \mid \exists r > 0, B_r(a) \subseteq S\}$$

Exercise. Show that:

- (1) For any nonempty set $S \subseteq \mathbb{R}^n$, $\text{Int}(S)$ is open.
- (2) A set S is open iff $\text{Int}(S) = S$.

Definition 2.1.8 ► Exterior of a set

Let $\phi \neq S \subseteq \mathbb{R}^n$. The exterior of S is,

$$\text{Ext}(S) = \{a \in \mathbb{R}^n \mid \exists r > 0, B_r(a) \cap S = \phi\}$$

Exercise. Show that $\text{Ext}(S) = \text{Int}(\mathbb{R}^n \setminus S)$.

Example 2.1.2

For $S = [0, 2] \setminus \{1\} = [0, 1) \cup (1, 2]$, $1 \notin \text{Ext}(S)$.

Definition 2.1.9 ► Boundary of a set

Let $\phi \neq S \subseteq \mathbb{R}^n$. The boundary of S is,

$$\partial S = \{a \in \mathbb{R}^n \mid \forall r > 0, B_r(a) \cap S \neq \phi \text{ and } B_r(a) \cap (\mathbb{R}^n \setminus S) \neq \phi\}$$

Example 2.1.3

For $S = [0, 1) \cup (1, 2] \cup \{5\}$, $\partial S = \{0, 1, 2, 5\}$ but the set of limit points is $\{0, 1, 2\}$.

Exercise. Show that:

- (1) S is open iff $S \cap \partial S = \phi$.
- (2) S is closed iff $S \supseteq \partial S$.
- (3) S is closed iff $S = \bar{S} =: S \cup \partial S = S \cup \{\text{Limit points of } S\}$
- (4) $\bar{S} = \text{Int}(S) \sqcup \partial S$. This gives the partition $\mathbb{R}^n = \text{Int}(S) \sqcup \partial S \sqcup \text{Ext}(S)$
- (5) ∂S is closed.
- (6) Let $\{\mathcal{O}_i\}_{i=1}^n \subseteq \mathcal{P}(\mathbb{R})$ and define $\mathcal{O} = \prod_{i=1}^n \mathcal{O}_i$. If $\mathcal{O}_i$'s are open (closed), $\mathcal{O}$ is open (closed).

2.2 Limits and Continuity

Recall the notion of limit in $\mathbb{R}$:

Suppose $f : (a, b) \setminus \{c\} \rightarrow \mathbb{R}$ is a function. We say $\lim_{x \rightarrow c} f$ exists if there is $\alpha \in \mathbb{R}$ such that $\forall \varepsilon > 0, \exists \delta > 0$ such that $x \in D_\delta(c) \implies |f(x) - \alpha| < \varepsilon$, that is, $f(D_\delta(c)) \subseteq B_\varepsilon(\alpha)$; in such a case we say that $\lim_{x \rightarrow c} f = \alpha$.

We now extend this to $\mathbb{R}^n$.

Definition 2.2.1 ► Limits in $\mathbb{R}^n$

Let $S \subseteq \mathbb{R}^n, a \in \{\text{Limit points of } S\}$ and, $f : S \setminus \{a\} \rightarrow \mathbb{R}^m$. We say that $\lim_{x \rightarrow a} f = b$ if for all $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) \in B_\varepsilon(b)$ for all $x \in D_\delta(a) \cap S$.
In other words, $\lim_{x \rightarrow a} f = b$ if for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|f(x) - b\| < \varepsilon \quad \forall x \in S, 0 < \|x - a\| < \delta$$

Definition 2.2.2 ► Continuity in $\mathbb{R}^n$

Let $S \subseteq \mathbb{R}^n, a \in S$ and, $f : S \rightarrow \mathbb{R}^m$. We say that f is continuous at a if for all $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) \in B_\varepsilon(f(a))$ for all $x \in B_\delta(a) \cap S$, that is, $\|f(x) - f(a)\| < \varepsilon$ for all $x \in S$ with $\|x - a\| < \delta$.

Note: Any function defined on S is vacuously continuous at an isolated point a by our definition.