Lecture 4

4.1 More examples of Continuous maps

Example 4.1.1

Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, \ (x,y) \neq (0,0) \\ 0, \ (x,y) = (0,0) \end{cases}$$

f is clearly continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ so we need only check the limit at $(0,0)$. We have,

$$0 \leq \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \frac{1}{2} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \frac{1}{2} \|(x,y)\|$$

By the squeeze theorem, we get $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$, and hence, f is continuous on $\mathbb{R}^2$.

Exercise. Show that any linear map is continuous. (Hint: Use that the norm is continuous.)

Example 4.1.2

Let $D = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\} = \Pi_2^{-1}(\mathbb{R} \setminus \{0\})$. Consider the function

$$f : D \to \mathbb{R}$$
$$f(x,y) = x \sin \frac{1}{y}$$

Clearly f is continuous on D. We also have,

$$0 \leq \left| x \sin \frac{1}{y} \right| \leq \|(x,y)\|$$

and hence, by the Squeeze theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

So, we can extend f to $(0,0)$ continuously by defining it to be 0.

4.2 Uniform Continuity

Definition 4.2.1 ▶ Uniform Continuity

Let $f : \mathcal{O}_n \to \mathbb{R}^m$, where $\mathcal{O}_n$ is open in $\mathbb{R}^n$. f is uniformly continuous if for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$f(x) \in B_\varepsilon(f(a)) \quad \forall\, x \in B_\delta(a) \cap \mathcal{O}_n, \quad \forall\, a \in \mathcal{O}_n$$
$$\Updownarrow$$
$$\|f(x) - f(a)\| < \varepsilon \quad \forall\, \|x - a\| < \delta,\ a, x \in \mathcal{O}_n$$

Exercise. Show that uniform continuity implies continuity.

4.3 Derivatives

We will use the following notation for the sake of brevity:

(1) $\mathcal{O}_n$ denotes an open set in $\mathbb{R}^n$, and we omit the n if it is clear from context.

(2) A function $f : \mathbb{R}^n \to \mathbb{R}^m$ has components $f = (f_1, f_2, \ldots, f_m)$

Recall the notion of derivative in $\mathbb{R}$:
Let $f : \mathcal{O}_1 \to \mathbb{R}$ be a function and $a \in \mathcal{O}_1$. Then f is differentiable at a if there is a real number $\alpha (= f'(a))$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \alpha \tag{1}$$

This clearly cannot be carried over verbatim to functions of several variables; we can't divide by a vector! To get a reasonable definition, we note that (1) is equivalent to,

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \alpha h}{h} = 0$$

Now, $h \mapsto \alpha h$ is a linear map and we already know $\{\text{linear maps on } \mathbb{R}\} \longleftrightarrow \mathbb{R}$. So the derivative $f'(a)$ is really a linear map! This leads to the following definition.

Definition 4.3.1 ▶ Derivative in several variables

Let $f : \mathcal{O}_n \to \mathbb{R}^m$ and $a \in \mathcal{O}_n$. f is differentiable at a if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Lh}{\|h\|} = 0$$
$$\Updownarrow$$
$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - Lh\|}{\|h\|} = 0$$
$$\Updownarrow$$
$$\lim_{x \to a} \frac{\|f(x) - f(a) - L(x-a)\|}{\|x - a\|} = 0$$

If f is differentiable at $a \in \mathcal{O}_n$, its derivative is denoted as $Df(a)$. We say f is differentiable on $\mathcal{O}_n$ if it is differentiable at all $a \in \mathcal{O}_n$.

Theorem 4.3.1

Let $f : \mathcal{O}_n \to \mathbb{R}^m$ be differentiable at $a \in \mathcal{O}_n$. The derivative $Df(a)$ is unique.

Proof. Let $L = Df(a)$ and L_1 be any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - L_1 h\|}{\|h\|} = 0$$

Suppose $L_1 \neq L$. Then, there is $h_0 \in \mathbb{R}^n$ such that $\|h_0\| = 1$ and $Lh_0 \neq L_1 h_0$. Consider the map $h : \mathbb{R} \to \mathbb{R}^n$ given by $h(t) = th_0$. Then, by the triangle inequality,

$$\frac{\|L(h(t)) - L_1(h(t))\|}{|t|} \leq \frac{\|f(a+h) - f(a) - L(h(t))\|}{\|h(t)\|} + \frac{\|f(a+h) - f(a) - L_1(h(t))\|}{\|h(t)\|}$$

Taking the limit as $t \to 0$, both terms on the right go to 0 by definition, and hence

$$\lim_{t\to 0} \frac{\|L(h(t)) - L_1(h(t))\|}{|t|} = 0 \implies \lim_{t\to 0} \frac{|t|\|Lh_0 - L_1 h_0\|}{|t|} = 0 \implies Lh_0 = L_1 h_1$$

which clearly contradicts the assumption. So, the derivative $Df(a)$ is unique. $\square$

4.4 Examples

Example 4.4.1

Consider $f : \mathcal{O}_n \to \mathbb{R}^m$ defined by $f(x) = c$. For any $a \in \mathcal{O}_n$,

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - Oh}{\|h\|} = 0$$

where O denotes the zero linear map. Hence, f is differentiable on $\mathcal{O}_n$ and $Df(a) = O$ for any $a \in \mathcal{O}_n$.

Example 4.4.2

Consider a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$. For all $a \in \mathbb{R}^n$,

$$\lim_{h\to 0} \frac{L(a+h) - La - Lh}{\|h\|} = 0$$

Hence, L is differentiable everywhere and $DL(a) = L$ for all $a \in \mathbb{R}^n$. This is as expected, as the best linear approximation of a linear map is itself.

At this point, we are faced with the problem of actually computing derivatives of non-trivial maps. A priori, it is not even clear if functions that are made of various differentiable functions of one variable, say $f(x, y, z) = (x^2 e^y z, y^3 \sin(xy) \cos z)$, are differentiable! We will perform a series of reductions that will answer such basic questions about differentiability of functions and even provide techniques to compute the derivatives.