

Lecture 8

8.1 A kind of converse of Theorem 7.2.1

As we have seen in previous lecture, the differentiability of a function gives an explicit expression for derivative with the existence of partials. In this lecture, we will prove a sufficient condition for differentiability based on its partials, which will be our final reduction for derivatives.

Theorem 8.1.1 (Final Reduction)

Let $f : \mathcal{O}_n \rightarrow \mathbb{R}^m$ and $a \in \mathcal{O}_n$. Suppose, all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exists on $\mathcal{O}_n$ and continuous at $a \in \mathcal{O}_n$. Then,

$$Df(a) = J_f(a) = \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{m \times n}$$

Proof. Without loss of generality, we take $a = (0, \dots, 0) \in \mathcal{O}_n$ and $m = 1$.

Let's do some back calculation: We already "know",

$$L = J_f(a) = (f_{x_1}(0) \quad \dots \quad f_{x_n}(0)) \quad \text{and} \quad Lh = \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}(0) \quad \forall h \in \mathbb{R}^n$$

which we use in the following claim,

Claim

$$\frac{1}{\|h\|} |f(h) - f(0) - Lh| \rightarrow 0 \text{ as } h \rightarrow 0$$

Proof. Simply calculating,

$$\frac{1}{\|h\|} |f(h) - f(0) - Lh| = \frac{1}{\|h\|} \left| f(h) - f(0) - \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}(0) \right|$$

For every i , we define, $\hat{h}_i = (h_1, \dots, h_i, \underbrace{0, \dots, 0}_{n-i}, 0)$ and $\hat{h}_0 = 0$. Then,

$$\begin{aligned} f(h) - f(0) &= (f(\hat{h}_1) - f(\hat{h}_0)) + (f(\hat{h}_2) - f(\hat{h}_1)) + \dots + (f(\hat{h}_n) - f(\hat{h}_{n-1})) \\ &= \sum_{i=1}^n (f(\hat{h}_i) - f(\hat{h}_{i-1})) \end{aligned}$$

which implies,

$$\begin{aligned} \left| f(h) - f(0) - \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}(0) \right| &= \left| \sum_{i=1}^n \left(f(\hat{h}_i) - f(\hat{h}_{i-1}) - h_i \frac{\partial f}{\partial x_i}(0) \right) \right| \\ &= \left| \sum_{i=1}^n \underbrace{h_i \frac{\partial f}{\partial x_i}(\hat{h}_{i-1} + c_i e_i)}_{\text{By MVT, as explained below}} - h_i \frac{\partial f}{\partial x_i}(0) \right| \end{aligned}$$

For a fixed h , we fix $i \in [n]$. And we consider the map, $\eta_i : (h_i - \epsilon, h_i + \epsilon) \rightarrow \mathbb{R}$

$$\begin{array}{c} \xrightarrow{\eta_i} \\ t \longmapsto \hat{h}_{i-1} + te_i \longmapsto f(\hat{h}_{i-1} + te_i) \end{array}$$

defined by, $\eta_i(t) = f(\hat{h}_{i-1} + te_i)$. Clearly, η_i is differentiable on $(0, h_i)$ and continuous on $[0, h_i]$. Then, by Mean Value Theorem,

$$\underbrace{\eta_i(h_i)}_{f(\hat{h}_i)} - \underbrace{\eta_i(0)}_{f(\hat{h}_{i-1})} = \eta'_i(c_i)h_i = f_{x_i}(\hat{h}_{i-1} + c_i e_i)h_i \quad (\text{for some } c_i \in (0, h_i))$$

Now, observe that, as $h \rightarrow 0$, $\hat{h}_{i-1} + c_i e_i \rightarrow 0$ which in turn implies, $f_{x_i}(\hat{h}_{i-1} + c_i e_i) \rightarrow f_{x_i}(0)$. Therefore,

$$\begin{aligned} \frac{1}{\|h\|} |f(h) - f(0) - Lh| &= \frac{1}{\|h\|} \left| \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}(\hat{h}_{i-1} + c_i e_i) - h_i \frac{\partial f}{\partial x_i}(0) \right| \\ &\leq \frac{1}{\|h\|} \sum_{i=1}^n |h_i| \left| \frac{\partial f}{\partial x_i}(\hat{h}_{i-1} + c_i e_i) - \frac{\partial f}{\partial x_i}(0) \right| \quad (\text{Triangle inequality}) \\ &\leq \sum_{i=1}^n \left| \frac{\partial f}{\partial x_i}(\hat{h}_{i-1} + c_i e_i) - \frac{\partial f}{\partial x_i}(0) \right| \quad (\text{as } \|h\| \geq |h_i| \forall i) \\ &\rightarrow 0 \text{ as } h \rightarrow 0 \end{aligned}$$

□

And, this completes the proof. □

With Theorem 8.1.1, computation of derivative is much easier when we are in favorable situation. Note that,

- (i) If f is differentiable at a then $\frac{\partial f_i}{\partial x_j}(a)$ exists for all i, j and $Df(a) = J_f(a)$.
- (ii) If $\frac{\partial f_i}{\partial x_j}$ is continuous at a then f is differentiable and $Df(a) = J_f(a)$.

The gap between (i) and (ii) is the continuity of partials, which is removable.

8.2 Examples

We conclude the lecture with some instructive examples.

Example 8.2.1 (Differentiable but discontinuous)

Take,

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right), & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases}$$

Then,

$$\begin{aligned} |f(x, y) - f(0, 0)| &= |x^2 + y^2| \left| \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \right| \\ &\leq (x^2 + y^2) = \|(x, y)\|^2 \end{aligned}$$

implies that f is continuous at $(0, 0)$. For all $(x, y) \neq (0, 0)$, the partials of f ,

$$\begin{aligned} f_x(x, y) &= 2x \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) - \frac{x}{\sqrt{x^2 + y^2}} \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \\ f_y(x, y) &= 2y \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) - \frac{y}{\sqrt{x^2 + y^2}} \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \end{aligned}$$

And at $(0, 0)$,

$$\begin{aligned} f_x(0, 0) &= \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = 0 \\ f_y(0, 0) &= \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = 0 \end{aligned}$$

Also,

$$\frac{1}{\sqrt{h^2 + k^2}} \left| f(h, k) - f(0, 0) - \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \right| = \sqrt{h^2 + k^2} \left| \sin\left(\frac{1}{\sqrt{h^2 + k^2}}\right) \right| \leq \|(h, k)\|$$

which shows that, f is differentiable at $(0, 0)$ and $Df(0, 0) = \begin{pmatrix} 0 & 0 \end{pmatrix}$. But, f_x and f_y are not continuous at $(0, 0)$!

So, even if a function is differentiable at some point, its partials may still not be continuous there!

Example 8.2.2 (Exercise)

Take,

$$f(x, y) = \begin{cases} x^{\frac{4}{3}} \sin\left(\frac{y}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

- Show that,
 1. f is differentiable on $\mathbb{R}^2$.
 2. f_x and f_y exist and continuous on $\mathcal{O}_2 = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$.
 3. f_x is not continuous at $(0, y)$ for all $y \neq 0$.
- Discuss the nature of continuity of f at the origin.

Example 8.2.3

Let, $f : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be defined by,

$$f(x, y) = (x + 2y + 3z, xyz, \cos x, \sin x)$$

Then, the at (x, y, z)

$$J_f(x, y, z) = \begin{pmatrix} 1 & 2 & 3 \\ yz & zx & xy \\ -\sin x & 0 & 0 \\ \cos x & 0 & 0 \end{pmatrix}$$

which has every entry continuous, thus

$$J_f(x, y, z) = Df(x, y, z)$$