

Lecture 16

16.1 Riemann-Darboux Integration

Definition 16.1.1 ► Volume of a closed/open ball

We define the volume of a closed ball $B^n = \prod_{i=1}^n [a_i, b_i]$ as $\text{Vol}(B^n) = \prod_{i=1}^n (b_i - a_i)$. We also define the volume of the open set $O^n = \prod_{i=1}^n (a_i, b_i)$ to be equal to that of B^n , i.e., $\text{Vol}(O^n) = \text{Vol}(B^n)$.

We now introduce some notation. Fix $i \in \{1, 2, \dots, n\}$. We define a partition of the i^{th} interval $[a_i, b_i]$ as

$$P_i : a_i = a_{i,0} < a_{i,1} < \dots < a_{i,n_i} = b_i$$

and the intervals of its partition as

$$I_{i,t} = [x_{i,t-1}, x_{i,t}], \forall 1 \leq t \leq n_i$$

and set

$$B_{(t_1, t_2, \dots, t_n)}^n = B_\alpha = [x_{1,t_1-1}, x_{1,t_1}] \times \dots \times [x_{n,t_n-1}, x_{n,t_n}] = I_{1,t_1} \times \dots \times I_{n,t_n}$$

where α is chosen from the indexing set $\Lambda(P) = \{\alpha = (t_1, \dots, t_n) \mid 1 \leq t_i \leq n_i, i = 1, \dots, n\}$.

Note

1. $B^n = \bigcup_{\alpha \in \Lambda(P)} B_\alpha^n$
2. $\text{Vol}(B^n) = \sum_{\alpha \in \Lambda(P)} \text{Vol}(B_\alpha^n)$

We call $\mathcal{P}(B) = \{P_1 \times \dots \times P_n \mid P_i \in \mathcal{P}[a_i, b_i]\}$ the set of all partitions of B^n .

Definition 16.1.2 ► Refinement of Partitions

Given $P = \prod_{i=1}^n P_i$ and $\tilde{P} = \prod_{i=1}^n \tilde{P}_i$ with $P, \tilde{P} \in \mathcal{P}[a, b]$, then $\tilde{P}$ is called a refinement of P if $\tilde{P}_i \supset P_i \forall i = 1, 2, \dots, n$.

Theorem 16.1.1

Let f be a bounded function over B^n . Let $P, \tilde{P} \in \mathcal{P}(B^n)$ and $\tilde{P} \supset P$. Then

$$L(f, P) \leq L(f, \tilde{P}) \leq U(f, \tilde{P}) \leq U(f, P)$$

Proof. Note that $L(f, \tilde{P}) \leq U(f, \tilde{P})$ follows directly from the fact that $m_\alpha(\tilde{P}) \leq M_\alpha(\tilde{P}) \forall \tilde{P} \in \mathcal{P}[a, b]$, where $m_\alpha = \inf_{B_\alpha^n} f$ and $M_\alpha = \sup_{B_\alpha^n} f$. $\square$

Corollary (Inequality of upper and lower sums)

For all $P, \tilde{P} \in \mathcal{P}(B^n)$, the following inequality holds.

$$m \times \text{Vol}(B^n) \leq L(f, P) \leq U(f, \tilde{P}) \leq M \times \text{Vol}(B^n)$$

We denote $\mathcal{B}(A) = \{f : A \rightarrow \mathbb{R} \mid \sup_A |f| < \infty\}$ as the set of all bounded functions over A for any $A \subseteq \mathbb{R}^n$.

Definition 16.1.3 ► Upper and Lower Darboux Integrals

For $f \in \mathcal{B}(B^n)$, we define

$$\overline{\int}_{B^n} f = \inf_{P \in \mathcal{P}(B^n)} U(f, P) \text{ and } \underline{\int}_{B^n} f = \sup_{P \in \mathcal{P}(B^n)} L(f, P)$$

as the Upper and Lower Darboux Integrals, respectively.

We have $L(f, P) \leq U(f, P')$ for all $P, P' \in \mathcal{P}(B^n)$ by taking the common refinement $\hat{P} = P \cup P'$. Hence,

$$\underline{\int}_{B^n} f \leq \overline{\int}_{B^n} f$$

Definition 16.1.4 ► Darboux Integral

Let $f \in \mathcal{B}(B^n)$. f is said to be Riemann-Darboux Integrable or Riemann Integrable or just Integrable if

$$\underline{\int}_{B^n} f = \overline{\int}_{B^n} f$$

In this case, we introduce the notation,

$$\int_{B^n} f \, dV = \int_{B^n} f(x_1, \dots, x_n) \, dx_1 \cdots dx_n = \underline{\int}_{B^n} f = \overline{\int}_{B^n} f$$

At this point, the notation $\int_{B^n} f(x_1, \dots, x_n) \, dx_1 \cdots dx_n$ **does not** indicate repeated integration, but we will see that it represents repeated integration for “nice” functions.