

Lecture 17

17.1 Properties of Riemann-Darboux Integration

In the previous lecture, we introduced the Riemann-Darboux Integral. In this lecture, we will explore some important properties of this integral, starting with a characterization.

Theorem 17.1.1 (Classification of Riemann Integrable Functions)

Let $f \in \mathcal{B}(B^n)$. Then $f \in \mathcal{R}(B^n)$ if and only if for every $\epsilon > 0$, there exists a partition $P \in \mathcal{P}(B^n)$ of B^n such that

$$(0 \leq) U(f, P) - L(f, P) < \epsilon$$

Proof. $\implies$ Suppose $f \in \mathcal{R}(B^n)$. Then we have

$$\overline{\int} f - \underline{\int} f = 0$$

Thus,

$$\begin{aligned} 0 = \overline{\int} f - \underline{\int} f &= \inf_{P \in \mathcal{P}(B^n)} U(f, P) - \sup_{P \in \mathcal{P}(B^n)} L(f, P) \\ &= \inf_{P \in \mathcal{P}(B^n)} (U(f, P) - L(f, P)) \end{aligned}$$

Hence for all $\epsilon > 0$, there exists a partition $P \in \mathcal{P}(B^n)$ such that $U(f, P) - L(f, P) < \epsilon$.

$\Leftarrow$ Conversely, assume that for every $\epsilon > 0$, there exists a partition $P \in \mathcal{P}(B^n)$ such that $U(f, P) - L(f, P) < \epsilon$. We want to show that $\overline{\int} f = \underline{\int} f$. Since $U(f, P) \geq \overline{\int} f$ and $L(f, P) \leq \underline{\int} f$, it follows that for all $\epsilon > 0$,

$$0 \leq \overline{\int} f - \underline{\int} f < \epsilon$$

This implies $\overline{\int} f = \underline{\int} f$, showing that $f \in \mathcal{R}(B^n)$. □

Exercise. Let $f, g \in \mathcal{R}(B^n)$. Then show that,

- $|f| \in \mathcal{R}(B^n)$ and $\left| \int_{B^n} f \right| \leq \int_{B^n} |f|$

- $\mathcal{R}(B^n)$ is a $\mathbb{R}$ -algebra by showing the following,

- (i) For any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in \mathcal{R}(B^n)$
- (ii) $fg \in \mathcal{R}(B^n)$

Next, we demonstrate the Riemann integrability of a class of “nice” functions (continuous). However, before proceeding, let’s introduce the concepts of the diameter of a set and the mesh of a partition.

Definition 17.1.1 ► Diameter of a set

For a set $A \subseteq \mathbb{R}^n$, the diameter $d(A)$ is defined as $d(A) = \sup\{|x - y| \mid x, y \in A\}$.

Exercise. Show that $d(B^n) = \max\{\|v_i - v_j\| \mid v_i, v_j \text{ are vertices of } B^n\}$

Definition 17.1.2 ► Mesh of a Partition

For a partition $P \in \mathcal{P}(B^n)$, the mesh $\|P\|$ is defined as $\|P\| = \max\{d(B_\alpha^n) \mid \alpha \in \Lambda(P)\}$.

Theorem 17.1.2 (Continuous Functions are Riemann Integrable)

The set of all continuous functions over B^n is contained in $\mathcal{R}(B^n)$, i.e.,

$$C(B^n) \subset \mathcal{R}(B^n)$$

Proof. Let $f \in C(B^n)$. Since f is uniformly continuous, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B^n$ with $\|x - y\| < \delta$, we have,

$$|f(x) - f(y)| < \underbrace{\frac{\epsilon}{2 \text{Vol}(B^n)}}_{\text{Call it } \tilde{\epsilon}} \quad (17.1)$$

Let $P \in \mathcal{P}(B^n)$ be a partition such that $\|P\| < \delta$. For each $\alpha \in \Lambda(P)$, let $a_\alpha \in B_\alpha^n$. Then $\|x - a_\alpha\| < \delta$ for all $x \in B_\alpha^n$. It follows from the uniform continuity condition (17.1) that,

$$\begin{aligned} |f(x) - f(a_\alpha)| &< \tilde{\epsilon} \\ \text{i.e., } f(a_\alpha) - \tilde{\epsilon} &< f(x) < f(a_\alpha) + \tilde{\epsilon} \end{aligned} \quad (17.2)$$

Since, (17.2) holds for all $\alpha \in \Lambda(P)$, $a_\alpha \in B_\alpha^n$ and for all $x \in B_\alpha^n$ we have,

$$f(a_\alpha) - \tilde{\epsilon} \leq m_\alpha \leq M_\alpha \leq f(a_\alpha) + \tilde{\epsilon}$$

Multiplying the volumes of B_α^n and summing over $\Lambda(P)$, we obtain,

$$\sum_{\alpha \in \Lambda(P)} f(a_\alpha) \text{Vol}(B_\alpha^n) - \frac{\epsilon}{2} \leq L(f, P) \leq U(f, P) \leq \sum_{\alpha \in \Lambda(P)} f(a_\alpha) \text{Vol}(B_\alpha^n) + \frac{\epsilon}{2}$$

Thus, $U(f, P) - L(f, P) < \epsilon$, and since ϵ is arbitrary, we conclude that $f \in \mathcal{R}(B^n)$. $\square$

Now, let’s consider an important question: Does an analogue of the Fundamental Theorem of Calculus exist in higher dimensions?

In one dimension ($n = 1$), we have the useful relationship $\int_{[a,b]} df = f|_{\partial[a,b]}$, which aids in computation.

However, this relationship becomes less practical in higher dimensions. For instance, in $n = 1$, the continuous counterpart to a sum $\sum a_n$ is the one-dimensional integral $\int_{B^1} f$. Similarly, in $n = 2$, the continuous analogue to a double sum $\sum a_{mn}$ is the two-dimensional integral $\int_{B^2} f$.

17.2 Iterated Integrals

Before delving deeper into the concept of integrability, let's take a brief detour to discuss the idea of a double sum.

Definition 17.2.1 ► Convergence of Double Sequence

A double sequence $\{a_{mn}\}$ converges to a if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|a_{mn} - a| < \epsilon$ for all $m, n \geq N$

Consider the following examples,

Example 17.2.1

- Let's take the sequence $\{a_{mn}\}$ defined by $a_{mn} = \frac{1}{m+n}$ for all $m, n \in \mathbb{N}$. This sequence is bounded, and for $N > \frac{1}{2\epsilon}$, we have $|a_{mn} - 0| = a_{mn} < \epsilon$ for all $m, n \geq N$.
- Now consider the sequence $\{a_{mn}\}$ defined as follows,

$$a_{mn} = \begin{cases} n & \text{if } m = 1 \\ \frac{1}{m+n} & \text{otherwise} \end{cases}$$

This sequence is also convergent but not bounded.

Recall the relation between total limit and iterated limits in double sequence,

Theorem

For a double sequence $\{a_{mn}\}$ if $\lim_{m,n \rightarrow \infty} a_{mn}$ exists and $\lim_{m \rightarrow \infty} a_{mn}$ exists for all n , then

$$\lim_{n \rightarrow \infty} \left(\lim_{m \rightarrow \infty} a_{mn} \right) = \lim_{m,n \rightarrow \infty} a_{mn}$$

An important analogue of the above result is Fubini's Theorem. Computation of the Darboux integral is typically a challenging task. However, Fubini's Theorem offers a valuable approach that simplifies the computation by utilizing iterated integrals.

Visualization

We look at slice functions along each axis, which enables us to simplify computations and apply Fubini's Theorem for efficient evaluation of multivariable integrals.

Consider a function $f : B^2 \rightarrow \mathbb{R}$. For each $x \in [a_1, b_1]$ we define a slice function $f_x : [a_2, b_2] \rightarrow \mathbb{R}$ given by $f_x(y) = f(x, y)$ for all $y \in [a_2, b_2]$. This function is obtained by fixing x and slicing along the y -axis at that x -coordinate. Then an iterated integral becomes

$$\int_{[a_1, b_1]} \left(\int_{[a_2, b_2]} f_x(y) dy \right) dx$$

The question arise whether this quantity is invariant under the interchange of x and y , i.e., we may slice f along x -axis at y to obtain $f_y : [a_1, b_1] \rightarrow \mathbb{R}$ given by $f_y(x) = f(x, y)$ for every $x \in [a_1, b_1]$ and want to investigate the equality of

$$\int_{[a_1, b_1]} \left(\int_{[a_2, b_2]} f_x(y) dy \right) dx \stackrel{?}{=} \int_{[a_2, b_2]} \left(\int_{[a_1, b_1]} f_y(x) dx \right) dy \stackrel{?}{=} \int_{B^2} f \quad (17.3)$$

In this context, we observe that a partition $P \in \mathcal{P}(B^2)$ can be decomposed into the partitions of the individual coordinates. Specifically, we have $P = P_1 \times P_2$ for the two coordinate intervals $[a_1, b_1]$ and $[a_2, b_2]$, and the corresponding indexing sets satisfy $\Lambda(P) = \Lambda(P_1) \times \Lambda(P_2)$.

Now consider the following example,

Example 17.2.2 (A discrepancy: Function integrable, slices not)

Let $I = [0, 1]$ and $B^2 = I \times I$. Consider the function $f : B^2 \rightarrow \mathbb{R}$ given by,

$$f(x, y) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, y \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

So the x -slice becomes,

$$f_x \equiv 0 \text{ for all } x \neq \frac{1}{2} \quad \text{and} \quad f_{\frac{1}{2}}(y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q}^c \cap [0, 1] \\ 0 & \text{if } y \in \mathbb{Q} \cap [0, 1] \end{cases}$$

Dirichlet Function

and y -slice,

$$\begin{aligned} \text{For } y \in \mathbb{Q} \cap [0, 1], \quad f_y(x) &= \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ \text{For } y \in \mathbb{Q}^c \cap [0, 1], \quad f_y &\equiv 0 \end{aligned}$$

Clearly, $f_x \in \mathcal{R}(I)$ for all $x \in \frac{1}{2}$ but $f_{\frac{1}{2}} \notin \mathcal{R}(I)$ and $f_y \in \mathcal{R}(I)$ for every y . So,

$$\int_I f_y = 0 \implies \int_I \left(\int_I f_y(x) \, dx \right) dy = 0$$

But, $\int_I f_x$ doesn't exist for $x = \frac{1}{2}$ which means $x \mapsto \int_I f_x$ is not a well-defined function on $[0, 1]$.

Hence, $\int_I \left(\int_I f_x(y) \, dy \right) dx$ doesn't exist. Yet $f \in \mathcal{R}(B^2)$. To see this, we fix $\epsilon > 0$ and consider the partition $P = P_1 \times P_2$ where,

$$\begin{cases} P_1 : 0 < \frac{1}{2} - \epsilon < \frac{1}{2} + \epsilon < 1 \\ P_2 : 0 < 1 \end{cases}$$

$$\text{So, } P = \left\{ \underbrace{\left[0, \frac{1}{2} - \epsilon\right] \times I}_{B_{\alpha_1}}, \underbrace{\left[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right] \times I}_{B_{\alpha_2}}, \underbrace{\left[\frac{1}{2} + \epsilon, 1\right] \times I}_{B_{\alpha_3}} \right\}.$$

Then $m_{\alpha_1} = m_{\alpha_2} = m_{\alpha_3} = 0$, $M_{\alpha_1} = M_{\alpha_3} = 0$ and $M_{\alpha_2} = 1$, which implies $U(f, P) - L(f, P) = 2\epsilon < 3\epsilon$. This shows that $f \in \mathcal{R}(B^2)$. Again $L(f, P) = 0$ for all $P \in \mathcal{P}(B^2)$ and hence,

$$\int_{B^2} f = 0$$

Question. Under which conditions does (17.3) hold?

Answer. Fubini's Theorem. The conditions for (17.3) to hold will be discussed in the next lecture.