

Lecture 23

We will begin this lecture with few examples.

Example 23.0.1 (Path of a Projectile)

$$\begin{aligned}\gamma(t) &= (\alpha t, \beta t - 16t^2) \\ \implies \gamma'(t) &= (\alpha, \beta - 32t) \\ \text{path length} &= \int \|\gamma'(t)\| dt \\ &= \int \sqrt{\alpha^2 + (\beta - 32t)^2} dt\end{aligned}$$

Example 23.0.2 (Perimeter of a Circle)

Parametrization of a circle of radius r is given by $\gamma(t) = (r \cos(t), r \sin(t)), t \in [0, 2\pi)$.

$$\begin{aligned}\gamma'(t) &= (-r \sin(t), r \cos(t)) \\ \ell(\gamma) &= \int_0^{2\pi} \|\gamma'(t)\| dt \\ &= \int_0^{2\pi} r dt \\ &= 2\pi r\end{aligned}$$

Example 23.0.3 (Arc Length of graph of functions)

Let, $f : [a, b] \rightarrow \mathbb{R}$ be a C^1 function. Consider $\gamma(t) = (t, f(t))$. It is a smooth curve.

$$\begin{aligned}\gamma'(t) &= (1, f'(t)) \\ \ell(\gamma) &= \int_a^b \|\gamma'(t)\| dt \\ &= \int_a^b \sqrt{1 + f'(t)^2} dt\end{aligned}$$

23.1 Line Integrals

To integrate a function over a curve we use **Line integral**. The function we should integrate maybe a **Scalar Field** or a **Vector Field**. (A quick example of a Vector Field: $f : \mathcal{O}_n \rightarrow \mathbb{R}$ be a differentiable function, then ∇f is a vector field.)

Question. Given a scalar field $f : \mathcal{O}_n \rightarrow \mathbb{R}$ and $\gamma \equiv \mathcal{C}$ be a curve, we want to define $\int_{\mathcal{C}} f$. But exactly how we can do this?

Answer. $\mathcal{C}$ is a curve, so it is bounded subset of $\mathbb{R}^n$. How about thinking of **Riemann Integration**? For $n \geq 2$, $\mathcal{C}$ is **content zero** in $\mathbb{R}^n$. This does not make any sense! The right way is as following.

Let, $\gamma : [a, b] \rightarrow \mathbb{R}^n$ be a **smooth curve** (or piecewise smooth) and $\mathcal{C} := \text{ran}(\gamma)$ (on other words path of γ). Let, $f \in \mathcal{B}(\mathcal{C})$. Given $\mathcal{P} \in \mathcal{P}[a, b]$, $\mathcal{P} : a = t_0 < t_1 < \dots < t_m = b$.

Let, $I_i = [t_{i-1}, t_i]$ be the sub-intervals and $\mathcal{C}_i = \gamma(I_i)$. Since, γ is smooth there is nice correspondence between I_i and $\mathcal{C}_i$. Also denote s_i by $\|\gamma(t_i) - \gamma(t_{i-1})\|$. As previous, define $m_i = \inf_{\mathcal{C}_i} f$ and $M_i = \sup_{\mathcal{C}_i} f$.

$$U(f, \mathcal{P}) = \sum_{i=1}^m M_i \cdot s_i$$

$$L(f, \mathcal{P}) = \sum_{i=1}^m m_i \cdot s_i$$

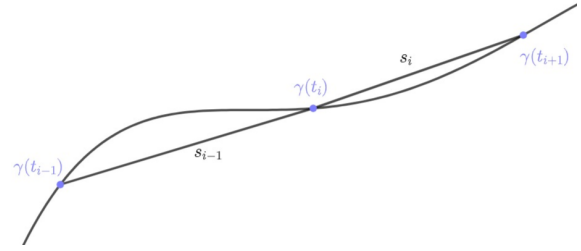


Figure 23.1: Curve $\mathcal{C}$

The above expressions are same as upper and lower Riemann sum respectively. This opens up “**The Pandora’s box!**”.

We can now use all the theory we used for the standard Riemann Integrals. We say f is line integrable over γ if,

$$\inf_{\mathcal{P} \in \mathcal{P}[a, b]} U(f, \mathcal{P}) = \sup_{\mathcal{P} \in \mathcal{P}[a, b]} L(f, \mathcal{P})$$

More over we will write the common value of the above equality as $\int_{\mathcal{C}} f$ and call this “The Line Integral over curve $\mathcal{C}$ ”.

Now we can invoke all theory we derived for 1 variable integration! We call $\mathcal{R}(\mathcal{C})$ the set of all Riemann integrable functions over $\mathcal{C}$.

Theorem 23.1.1

Let, γ be a “Rectifiable” smooth(or piecewise smooth) and $\mathcal{C} = \text{ran}(\gamma)$ and $f \in \mathcal{B}(\mathcal{C})$. Then,

$$1. f \in C^0(\mathcal{C}) \implies f \in \mathcal{R}(\mathcal{C})$$

2.

$$f \in \mathcal{R}(\mathcal{C}) \iff \lim_{\|\mathcal{P}\| \rightarrow 0} \sum_{i=1}^m f(\zeta_i) s_i \text{ exist and equal to } \int_{\mathcal{C}} f.$$

Here, ζ_i is tag of the interval I_i .

3. (This requires smoothness) If γ is C^1 and smooth, $f \in \mathcal{R}(\mathcal{C})$, then

$$\int_{\mathcal{C}} f = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt$$

Proof. **Exercise.**

In the above theorem equation in 3 is also independent of choice of Parametrization of γ . Because any other smooth parametrized curve $\tilde{\gamma} = \gamma \circ \varphi$ where, φ is an onto continuous function.

Facts: $\mathcal{C}$ be a piecewise smooth, parametrized curve γ . $f, g \in \mathcal{R}(\mathcal{C})$ and $r \in \mathbb{R}$, then

- $\int f + rg = \int f + r \int g$
- $f \geq g$ over $\mathcal{C}$ then $\int f \geq \int g$
- $\int |f| \geq \left| \int f \right|$
- If $a < d < b$, if $\gamma_1 := \gamma|_{[a,d]}$ and $\gamma_2 := \gamma|_{[d,b]}$ then

$$\int_{\mathcal{C}} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

We have resolved the problems for Scalar field. **What about vector fields?**

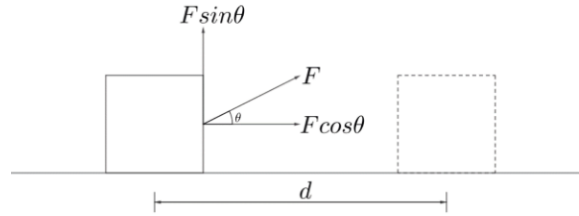


Figure 23.2: Work done by a constant Force

Suppose a particle moves a distance d under a constant force F , then work done by the force is $Fd \cos \theta = \vec{F} \cdot \vec{d}$.

If the force was not constant throughout the path then how can we calculate work done by that force? Consider the case where F is the vector field (Force in this case) defined over a curve (path) γ . Here, $\gamma : [a, b] \rightarrow \mathbb{R}^n$ and $\mathcal{C} = \text{ran}(\gamma)$. So, work done throughout the whole path will be,

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

Which is equal to, $\int_a^b \vec{F}(\gamma(t)) \cdot \nabla \gamma(t) dt$

Now we will look into some examples.

Example 23.1.1

Find work done by the field $F(x, y, z) = (xy, xz, yz)$ along the curve $\gamma(t) = (t^2, -t^3, t^4), t \in [0, 1]$.

Answer.

$$\gamma'(t) = (2t, -3t^2, 4t^3)$$

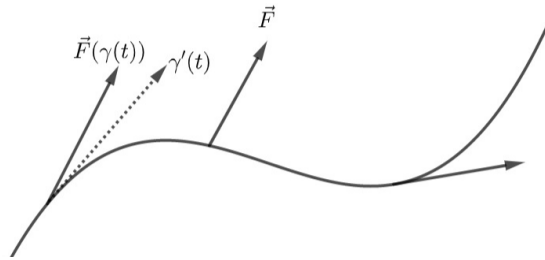


Figure 23.3: $\vec{F}$ is the vector field over the curve γ

$$\begin{aligned}\implies \text{Work done} &= \int_0^1 (-t^5, t^6, -t^7) \cdot (2t, -3t^2, 4t^3) dt \\ &= -\frac{31}{88}\end{aligned}$$

23.1.1 Line Integration of a Vector Field

$F : \mathcal{O}_n \rightarrow \mathbb{R}^n$ be a vector field and $\gamma : [a, b] \rightarrow \mathcal{O}_n$ be a curve. We consider a partition $\mathcal{P} : a = t_0 < t_1 < \dots < t_m = b$, Let $\mathcal{C}_i = \gamma|_{[t_{i-1}, t_i]}$ and $\gamma_i = \gamma(t_i)$, $\Delta r_i = \gamma_i - \gamma_{i-1}$.

$$R(F; \mathcal{P}) = \sum_{i=1}^m F(\gamma_i) \cdot \Delta r_i$$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \lim_{\|\mathcal{P}\| \rightarrow 0} R(F, \mathcal{P}) \quad (\text{if the limit exists})$$

Just like the scalar field, if γ is C^1 and smooth, then

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt \quad (23.1)$$