Lecture 24

Theorem 24.0.1

Let, $f : \mathcal{O}_n \to \mathbb{R}$ be a C^1 function and γ be a piecewise smooth C^1 curve on $\mathcal{O}_n$, joining two points A and B. Then

$$\int_c \boldsymbol{\nabla} f \cdot \mathrm{d}\vec{r} = f(B) - f(A)$$

Which means the above line integral is "independent of parametrization".

Proof.
Let, r be a parametrization of the curve γ and $r : [a, b] \to \mathcal{O}_n$ such that $r(a) = A$ and $r(b) = B$.
Evaluating the LHS gives,

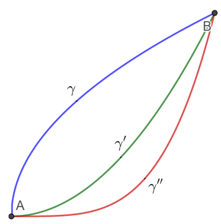

Figure 24.1: Paths from A to B

$$\begin{aligned}
\int_c \boldsymbol{\nabla} f \cdot \mathrm{d}\vec{r} &\overset{(23.1)}{=} \int_a^b \boldsymbol{\nabla} f(r(t)) \cdot r'(t)\ \mathrm{d}t \\
&= \int_a^b \left(\sum_{i=1}^{n} \frac{\partial f}{\partial r_i} \times r_i'(t) \right) \mathrm{d}t \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f(r(t)))\ \mathrm{d}t \\
&= f(r(b)) - f(r(a)) \\
&= f(B) - f(A)
\end{aligned}$$

□

This result has an important consequence that we often use in Physics.

> "Work done by a conservative force always depends on the starting point and the end point, not on the path followed by the particle."

We know if the force is conservative then we can define potential energy U as $\vec{F} = -\boldsymbol{\nabla} U$. Work done by the force is simply the change of potential.

Now we will recall the basics of the Planes and Normals.

24.1 Planes and Normals

Let, $\vec{P_0} = \langle x_0, y_0, z_0 \rangle$ be a fixed vector in $\mathbb{R}^3$. $\vec{N} = \langle a, b, c \rangle \neq \vec{0}$. The plane through $\vec{P_0}$ with $\vec{N}$ as normal to this plane is,

$$\left\{ \vec{P_0} + \vec{P} \mid \vec{P} \cdot \vec{N} = 0 \right\} = \left\{ \vec{r} \mid (\vec{r} - \vec{P_0}) \cdot \vec{N} = 0 \right\}$$

<u>**Equation of the plane**</u>: Consider an arbitrary point $\vec{P} = \langle x, y, z \rangle$ on the plane then,

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

So, equation of the plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{24.1}$$

Let $\vec{Q_1}, \vec{Q_2}$ be independent in $\mathbb{R}^3$ and also satisfying $\vec{Q_i} \cdot \vec{N} = 0$. Clearly, $\vec{Q_1} \times \vec{Q_2} \neq \vec{0}$. We can see $(\vec{Q_1} \times \vec{Q_2}) \cdot \vec{Q_i} = 0$, i.e., $\{\vec{Q_1}, \vec{Q_2}, \vec{Q_1} \times \vec{Q_2}\}$ form a basis of $\mathbb{R}^3$.

$$\therefore \quad \vec{Q_1} \times \vec{Q_2} = c\vec{N}$$

So, $\left\{\vec{P_0} + r_1\vec{Q_1} + r_2\vec{Q_2} \mid r_1, r_2 \in \mathbb{R}\right\}$ describes the same plane as (24.1).

24.2 Surface and Surface Integrals

Definition 24.2.1 ▶ Region

A subset $\mathcal{R} \subseteq \mathbb{R}^2$ is called a "Region" if $\mathcal{R}$ is Open ans $\mathcal{R}$ has an area (i.e. $\partial\mathcal{R}$ is **content zero**)

Definition 24.2.2 ▶ Parametrized Surface

Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a region. A C^1 function $r : \mathcal{R} \to \mathbb{R}^3$ said to be a "Parametrized Surface" if :

- The component functions r_i have bounded partials
- r is $1 - 1$ function
- $\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v} \neq \vec{0}$ for all $(u, v) \in \mathbb{R}^2$. This means total derivative of r has rank 2.

We will call range of r as a **Surface**, $\mathcal{S} = \text{ran}(r)$.

Let, η be a map defined on $(-\varepsilon, \varepsilon)$ that maps $t \xmapsto{\eta} r(u_o = 0 + t, v_0)$. Clearly, η defines smooth curve on $\mathcal{S}$. Similarly, we can define $\tilde{\eta}$ on $(-\varepsilon, \varepsilon)$ that maps $t \xmapsto{\tilde{\eta}} r(u_0, v_0 + t)$.
Thus, $\tilde{\eta}$ also defines a smooth curve on $\mathcal{S}$.
From the Figure 24.2 we can see η and $\tilde{\eta}$ are the curves. $r_u(u_0, v_0) = \frac{\mathrm{d}\eta}{\mathrm{d}t}\Big|_{t=0}$, which gives the tangent of the curve η at point (u_0, v_0) along x axis.
Similarly, for $\tilde{\eta}$, $r_v(u_0, v_0)$ gives the tangent of $\tilde{\eta}$ along y axis. The vectors $r_u(u_0, v_0), r_v(u_0, v_0)$ spanned together to form a plane. This plane is known as **Tangent Plane**.
Since r is C^1, both $\vec{r_u}$ and $\vec{r_v}$ are continuous and hence $\vec{r_u} \times \vec{r_v}$ is continuous. Also, $\vec{r_u} \times \vec{r_v}$ is along the normal vector of $\mathcal{S}$ at $r(u_0, v_0)$ which follows from the previous section.

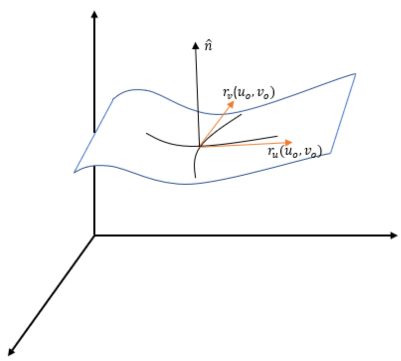

Figure 24.2

Now we will move towards a very important definition.

Definition 24.2.3 ▶ Tangent Plane

For a Parametrized Surface r, let $\operatorname{ran}(r) = \mathcal{S}$ and $r(u_0, v_0) = P$, then the plane generated by $r_u(u_0, v_0)$ and $r_v(u_0, v_0)$ through $r(u_0, v_0)$ is called the **Tangent Plane** of $\mathcal{S}$ through P.
We denote it by $T_P\mathcal{S}$.
Every element of $T_P\mathcal{S}$ is called **Tangent Vectors** at P on $\mathcal{S}$.

It can be shown that $T_P\mathcal{S}$ is independent of parametrization of $\mathcal{S}$, i.e., $T_P\mathcal{S}$ is independent of r (**Exercise**). Actually, for different $\tilde{r}$ of $\mathcal{S}$ the basis for $T_P\mathcal{S}$ will be changed. But they still generate the same plane.

24.3 Examples

We will go through some examples.

Example 24.3.1 (Graph of Function)

Let $f : \mathcal{O}_2 \to \mathbb{R}$ be a C^1 function then the graph of f is $\mathcal{G}(f) = \{(x, y, f(x, y)) : (x, y) \in \mathcal{O}_2\}$. Under the conditions $\mathcal{O}_2$ is bounded and partial derivatives of f is bounded, we want to find a parametrization of this Surface $\mathcal{G}(f)$.
Answer. Here, we use the trivial parametrization $r : \mathcal{O}_2 \to \mathbb{R}^3$ that is $r(x, y) = (x, y, f(x, y))$. Clearly, r is one - one. Now, $r_u(u, v) = (1, 0, f_u)$ and $r_v(u, v) = (0, 1, f_v)$.
So, $r_u \times r_v = (-f_u, -f_v, 1) \neq \vec{0}$. So, it is a parametrization of the surface.

Example 24.3.2 (Torus)

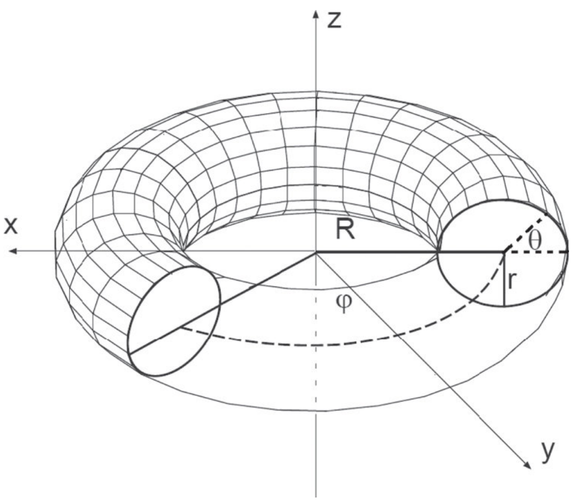

Figure 24.3: Torus.

Parametrization is given by, $(0 < r < R)$

$$r(\theta, \varphi) = ((R + r\cos\theta)\cos\varphi, (R + r\cos\theta)\sin\varphi, r\sin\theta); 0 \leq \theta, \varphi \leq 2\pi \tag{24.2}$$

Example 24.3.3 (Surface of Revolution)

Let f, g be C^1 functions on $[0, b]$. Consider the curve $t \longmapsto (0, f(t), g(t))$ is a C^1 curve. If we rotate the curve with respect to z axis, we must get a surface. Parametrization of the surface is given by,

$$r(u, v) = (f(u)\cos v, f(u)\sin v, g(u)) \quad ; u \in [0, b], v \in [0, 2\pi]$$

Exercise. Find the Parametrization of a sphere of radius R.