Lecture 28

28.1 Green's Theorem

Theorem 28.1.1 ($\mathbb{R}^2$ version of Green's Theorem)

Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a simply connected domain with boundary curve $\mathcal{C}$ where parametrization is taken in anti-clockwise direction. Let $\vec{F} = (P, Q)$ be a C^1 vector field on $\mathcal{R}$, then

$$\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}r := \int_{\mathcal{C}} P \,\mathrm{d}x + Q \,\mathrm{d}y = \int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A$$

Proof.
(for Simple region)
Let $\mathcal{R} = \{(x, y) \mid a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$ be a simple region. Here $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{V}_2 \cup \mathcal{C}_2 \cup \mathcal{V}_1$ is the curve bounding the region along anti-clockwise direction (as shown in Figure 28.1).

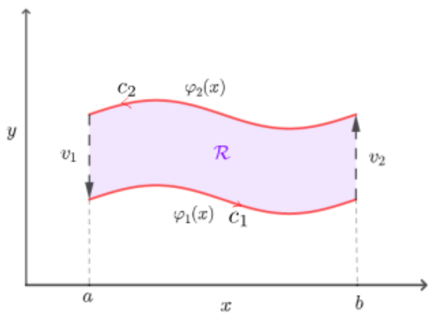


Figure 28.1: A simple region

Now,

$$-\int_{\mathcal{R}} \frac{\partial P}{\partial y} \,\mathrm{d}A = -\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y} \,\mathrm{d}y \,\mathrm{d}x$$
$$= -\int_a^b \left(P(x, \varphi_2(x)) - P(x, \varphi_1(x))\right) \mathrm{d}x$$

The curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{V}_1, \mathcal{V}_2$ can be explicitly written as,

$$\begin{aligned}
\mathcal{V}_1 &= \{(a, t) \mid \varphi_1(a) \leq t \leq \varphi_2(a)\} \\
\mathcal{C}_1 &= \{(x, \varphi_1(x)) \mid a \leq x \leq b\} \\
\mathcal{V}_2 &= \{(a, t) \mid \varphi_1(b) \leq t \leq \varphi_2(b)\} \\
\mathcal{C}_2 &= \{(x, \varphi_2(x)) \mid a \leq x \leq b\}
\end{aligned}$$

We compute the integrals for P over these curves and obtain,

$$\begin{aligned}
\int_{\mathcal{V}_1} P \,\mathrm{d}x &= \int_{\mathcal{V}_1} P(x(t), y(t)) \frac{\mathrm{d}x(t)}{\mathrm{d}t} \mathrm{d}t = 0 \\
\int_{\mathcal{C}_1} P \,\mathrm{d}x &= \int_a^b P(t, \varphi_1(t)) \,\mathrm{d}t \\
\int_{\mathcal{C}_2} P \,\mathrm{d}x &= \int_a^b P(t, \varphi_2(t)) \,\mathrm{d}t
\end{aligned}$$

$$\implies \int_{\mathcal{C}} P \mathrm{d}x = -\int_{\mathcal{R}} \frac{\partial P}{\partial y} \,\mathrm{d}A$$

By similar mechanism we can show $\displaystyle\int_{\mathcal{C}} Q\,\mathrm{d}y = \int_{\mathcal{R}} \frac{\partial Q}{\partial y}\,\mathrm{d}A$. The rest follows from here. $\square$

Example 28.1.1

Let $\mathcal{C}$ be the boundary of $[0,1]^2$, i.e., $\partial[0,1]\times[0,1] = \mathcal{C}$. Evaluate

$$\int_{\mathcal{C}} \langle x^2 - y^2, 2xy\rangle$$

Solution. We can decompose $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ (as in the following picture)
Let $P(x,y) = x^2 - y^2, Q(x,y) = 2xy$. Then the integral,

$$\begin{aligned}\int_{\mathcal{C}} P\,\mathrm{d}x + Q\,\mathrm{d}y &= \iint_{[0,1]^2} (2y+2y)\,\mathrm{d}A \qquad \text{(Green's Theorem)}\\ &= \int_0^1\int_0^1 4y\,\mathrm{d}y\,\mathrm{d}x\\ &= 2\end{aligned}$$

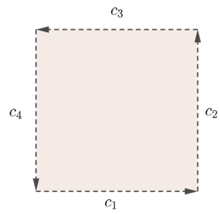


Figure 28.2: $\partial([0,1]^2)$

If we try to calculate the integral **directly**, we will end up getting same result.

Area of a closed Region. Let $\mathcal{R}$ (simply connected) be a closed region and $\mathcal{C} = \partial\mathcal{R}$ be the curve enclosing the region. Using Green's Theorem we get,

$$\text{Area}(\mathcal{R}) = \int_{\mathcal{R}} \mathrm{d}A = \int_{\mathcal{C}} x\mathrm{d}y = \int_{\mathcal{C}} -y\mathrm{d}x = \int_{\mathcal{C}} \frac{x\mathrm{d}y - y\mathrm{d}x}{2}$$

Example 28.1.2 (Area inside the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$)

Solution. Parametrization of ellipse $x = a\cos t, y = b\sin t$ where $t \in [0, 2\pi)$. Using the above application of Green's Theorem we can write,

$$\text{Area} = \int_{\mathcal{C}} x\mathrm{d}y = ab\int_0^{2\pi} \cos^2 t\mathrm{d}t = \pi ab$$

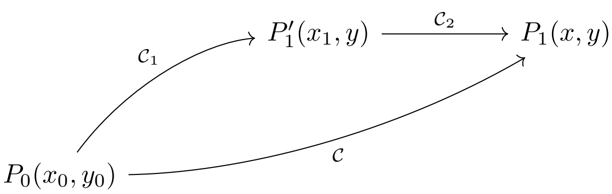
Theorem 28.1.2 (Independence of path)

Let $\vec{F}$ be a C^1 vector field on $\mathbb{R}^2$ such that $\displaystyle\int_{\mathcal{C}} \vec{F}\cdot \mathrm{d}\vec{r}$ is independent of path. Then $\vec{F}$ is conservative over an open and simply connected domain.

Proof. Let $\mathcal{D}$ be an open and connected domain. $\vec{F} = \langle P, Q\rangle$ is defined over $\mathcal{D}$. Also let $P_0 = \langle x_0, y_0\rangle$ be a fixed point in the domain $\mathcal{D}$ and $P_1 = \langle x, y\rangle \in \mathcal{D}$ be a variable point. $\mathcal{C}$ be a smooth curve joining P_0 and P_1. Define

$$\varphi(x,y) = \int_{\mathcal{C}} \vec{F}\cdot\mathrm{d}\vec{r}$$

Since, $\mathcal{D}$ is open set, so we must get an open ball centered at P_1 contained in $\mathcal{D}$. Take a point $P_1' = \langle x_1, y\rangle$ inside that open ball such that $x_1 < x$. Let $\mathcal{C}_1$ be a smooth curve from P_0 to P_1 and $\mathcal{C}_2$ be a line segment from P_1' to P_1. So, $\mathcal{C}_1 \cup \mathcal{C}_2$ defines a smooth curve from P_0 to P_1.



As $\displaystyle\int_{\mathcal{C}} \vec{F}\cdot \mathrm{d}\vec{r}$ is path independent We can write,

$$\varphi(x,y) = \int_{\mathcal{C}_1} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{\mathcal{C}_2} \vec{F}\cdot \mathrm{d}\vec{r}$$

Now we take the partial derivative of both sides of this equation with respect to x. The first integral does not depend on the variable x since $\mathcal{C}_1$ is the path from $P_0(x_0, y_0, z_0)$ to $P_1'(x_1, y, z)$ and so partial differentiating this line integral with respect to x is zero.

$$\begin{aligned}\frac{\partial \varphi}{\partial x} &= \frac{\partial}{\partial x}\left(\int_{\mathcal{C}_1} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{\mathcal{C}_2} \vec{F}\cdot \mathrm{d}\vec{r}\right)\\ &= \underbrace{\frac{\partial}{\partial x}\left(\int_{\mathcal{C}_1} \vec{F}\cdot \mathrm{d}\vec{r}\right)}_{=0} + \frac{\partial}{\partial x}\left(\int_{\mathcal{C}_2} \vec{F}\cdot \mathrm{d}\vec{r}\right)\end{aligned}$$

Also, $\mathcal{C}_2$ can be parametrized as $r(t) = \langle t, y\rangle$ where $t \in [x_1, x]$. So,

$$\begin{aligned}\frac{\partial}{\partial x}\left(\int_{\mathcal{C}_2} \vec{F}\cdot \mathrm{d}\vec{r}\right) &= \frac{\partial}{\partial x}\left(\int_{x_1}^{x} \langle P(t,y), Q(t,y)\rangle \cdot \langle 1, 0\rangle \,\mathrm{d}t\right)\\ &= \frac{\partial}{\partial x}\left(\int_{x_1}^{x} P(t,y)\ \mathrm{d}t\right)\\ &= P(x,y) \qquad \text{[Fundamental Theorem of calculus]}\end{aligned}$$

Similarly, we can show that, $\frac{\partial \varphi}{\partial y} = Q(x,y)$. And hence, $\boldsymbol{\nabla}\varphi = \vec{F}(x,y)$. We can define φ as the potential of $\vec{F}$. □

Theorem 28.1.3

Let $\mathcal{D}$ be a simply connected domain in $\mathbb{R}^2$ and $\vec{F}$ is a C^1 vector field on $\mathcal{D}$. Then $\vec{F}$ is conservative iff $\boldsymbol{\nabla}\times\vec{F} = 0$ on $\mathcal{D}$.

Proof. ($\Rightarrow$) This direction is trivial.

($\Leftarrow$) From Green's Theorem we can say that $\displaystyle\int_{\mathcal{C}} \vec{F}\cdot \mathrm{d}\vec{r} = 0$ over all closed curve $\mathcal{C}$. For any two point $p_0, p_1 \in \mathcal{D}$ if $\gamma_1, \gamma_2 : [0,1] \to \mathcal{D}$ are two smooth curves joining p_0 and p_1. (i.e., $\gamma_1(0) = \gamma_2(0) = p_0$ and $\gamma_1(1) = \gamma_2(1) = p_1$) then $\gamma_1 \cup \gamma_2(1-t)$ is a closed curve. So, $\displaystyle\int_{\gamma_1} \vec{F}\cdot \mathrm{d}\vec{r} = \int_{\gamma_2} \vec{F}\cdot \mathrm{d}\vec{r}$. Which means the integral is path independent. Using the previous theorem we can say, $\vec{F}$ is conservative on $\mathcal{D}$. □

28.2 Gauss Divergence Theorem

Definition 28.2.1 ▶ Divergence of a vector field

Given a vector field $\vec{F} = (f_1, \cdots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$, the "Divergence" of $\vec{F}$ is,

$$\operatorname{div}(F) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \equiv \boldsymbol{\nabla} \cdot \vec{F}$$

Theorem 28.2.1 (Gauss Divergence Theorem)

Let $\mathcal{D} \subseteq \mathbb{R}^3$ a solid domain, $\partial\mathcal{D}$ be an oriented surface. Let $\vec{F} = \langle P, Q, R\rangle$ be a C^1 vector field on an open surface containing $\mathcal{D} \cup \partial\mathcal{D}$. Then,

$$\underbrace{\int_{\partial\mathcal{D}=\mathcal{S}} \vec{F} \cdot \mathrm{d}\vec{S}}_{\text{surface integral}} = \underbrace{\int_{\mathcal{D}} \boldsymbol{\nabla} \cdot \vec{F} \,\mathrm{d}V}_{\text{volume integral}}$$

Just like FTC, the behavior over a volume is fully determined by the behavior at the boundary. Proof of this theorem is beyond our reach. But we can see the proof for simple cases.

Proof. (For a simple case) Consider $\mathcal{D} = \{(x, y, z) \mid \varphi_1(x, y) \le z \le \varphi_2(x, y), (x, y) \in [a, b] \times [c, d]\}$. (**Exercise.**) Complete the proof! □

Example 28.2.1

$F(x, y, z) = \langle x + y, z^2, x^2\rangle$ and S be the hemisphere $x^2 + y^2 + z^2 = 1, z > 0$. Compute,

$$\int_S \vec{F} \,\mathrm{d}\vec{S}$$

Solution. Notice that S is open surface. We want to use Gauss Theorem 28.2.1. So we need a close surface. Let S_1 be the surface $x^2 + y^2 \le 1$. Then $S \sqcup S_1$ is a closed surface.

$$\begin{aligned}
\int_{S\sqcup S_1} \vec{F} \cdot \mathrm{d}\vec{S} &= \int_{x^2+y^2,z^2\le 1,z\ge 0} \boldsymbol{\nabla} \cdot \vec{F} \,\mathrm{d}V \\
&= \int_{x^2+y^2,z^2\le 1,z\ge 0} \mathrm{d}V \\
&= \frac{2\pi}{3}
\end{aligned}$$

Parametrization of the surface $S_1 = \{(x, y, 0) \mid x^2 + y^2 = 1\}$. So, $r_x \times r_y = \langle 1, 0, 0\rangle \times \langle 0, 1, 0\rangle$.

$$\begin{aligned}
\int_{S_1} \vec{F} \cdot \mathrm{d}\vec{S} &= \int_{x^2+y^2\le 1} \langle x + y, z^2, x^2\rangle \cdot \langle 0, 0, 1\rangle \,\mathrm{d}A = \int_{x^2+y^2\le 1} x^2 \,\mathrm{d}A \\
&= \int_0^{2\pi} \int_0^1 r^3 \cos^2\theta \,\mathrm{d}r \,\mathrm{d}\theta = \frac{\pi}{4} \\
\Rightarrow \int_S \vec{F} \cdot \mathrm{d}\vec{S} &= \frac{11\pi}{12}
\end{aligned}$$

28.3 Stokes' Theorem

Theorem 28.3.1 (Stokes' Theorem)

Let $\mathcal{C}$ be a C^1 curve enclosing an oriented surface $\mathcal{S}$ in $\mathbb{R}^3$. Let, $\vec{F} = \langle P, Q, R\rangle$ be a C^1 vector field on an open set containing $\mathcal{S}$. Then,

$$\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r} = \int_{\mathcal{S}} \left(\boldsymbol{\nabla} \times \vec{F}\right) \cdot \mathrm{d}\vec{S}$$

Here orientation of $\mathcal{S}$ and direction of $\mathcal{C}$ is same.

Example 28.3.1

Compute $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r}$, where $\mathcal{C} : x^2 + y^2 = 9, z = 4$ and $\vec{F} = \langle -y, x, xyz\rangle$.

Solution. $\boldsymbol{\nabla} \times \vec{F} = \langle xz, -yz,, 2\rangle$. By convention, we should assume direction of $\mathcal{C}$ is along counter-clockwise direction. So, The normal vector of $\mathcal{S}$ is along negative z axis. So, required integral,

$$\begin{aligned}
\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r} &= \int_{\mathcal{S}} (\boldsymbol{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{S} \\
&= \int_{x^2+y^2\leq 1, z=4} (\boldsymbol{\nabla} \times \vec{F}) \cdot \langle 0, 0, -1\rangle \,\mathrm{d}A \\
&= -2 \int_{x^2+y^2\leq 1, z=4} \mathrm{d}A \\
&= -18\pi
\end{aligned}$$

Stoke's Theorem is the $\mathbb{R}^3-$analogue of Green's Theorem 28.1.1. If we take the third component of $\vec{F}$ to be zero, i.e., $R = 0$, then Stoke's Theorem 28.3.1 gives us back Green's Theorem 28.1.1.

There is a generalized version of Stokes' theorem. Just for information the theorem is stated below.
- If Ω is an oriented n-manifold (with boundary) and ω is a differential form ($(n-1)$ form). Then integral of ω over the boundary $\partial\Omega$ of the manifold Ω is given by,

$$\int_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}\omega$$