Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis -I

Home Assignment I — Due Date : 20 October 2021
Instructor: B V Rajarama Bhat

(1) Write down your favorite mathematical puzzle along with its solution.
(2) Write down your favorite theorem.
(3) Write down an open problem which you would like to work on.
(4) For $n \in \mathbb{N}$, take $A_n = \{1, 2, \dots, n\}$. Show that for $n > 1$, there exists a function $g : A_n \to A_n$, such that there does not exist any $f : A_n \to A_n$, satisfying $g = f \circ f$, that is, for $n > 1$, there exist functions on A_n without square roots.
(5) Show that any subset of a countable set is countable.
(6) Show that countable union of a countable set is countable, that is, if $\{A_i : i \in I\}$ is a family of countable sets, where the 'indexing set' I is also countable, then $\bigcup_{i\in I} A_i$ is countable.
(7) Show that the set $P = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_0, a_1, \dots, a_n \in \mathbb{Z}, n \in \mathbb{N}\bigcup\{0\}\}$ of polynomials with integer coefficients is countable.
(8) Use mathematical induction to prove the Binomial theorem: Let a, b be real numbers. Then

$$(a + b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^k b^{n-k}, \ \forall n \in \mathbb{N}.$$

(9) Consider the set $\mathbb{Z}_5 := \{0, 1, 2, 3, 4\}$ with binary operations $\oplus$ and $\odot$, which are addition and multiplication modulo 5, respectively. That is, $i \oplus j$ is the remainder, on dividing $i + j$ by 5. Similarly $i \odot j$ is the remainder, on dividing $i.j$ by 5. Show that $(\mathbb{Z}_5, \oplus, \odot)$ satisfies all the algebraic axioms of real numbers, but it does not satisfy the order axioms for any subset $\mathbb{P}$ of $\mathbb{Z}_5$.
(10) Consider the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, with following binary operations:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \ (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2;$$
$$(x_1, y_1).(x_2, y_2) = (x_1x_2, y_1y_2), \ (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2;$$

and zero element as $(0, 0)$ and 'one' as $(1, 1)$. Further, take the set of positive numbers as

$$\mathbb{P} = \{(x_1, y_1) : x_1 > 0, \text{ and } y_1 > 0\}.$$

Verify as to which of the axioms of real numbers are satisfied and which are not satisfied.