Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis -I

Home Assignment II — Due Date : 30 November 2021
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(1) Let A, B be bounded subsets of $\mathbb{R}$. Prove or disprove the following:

$$\sup(A\bigcup B) = \max\{\sup(A), \sup(B)\}.$$
$$\inf(A\bigcup B) = \min\{\inf(A), \inf(B)\}.$$
$$\sup(A\bigcap B) = \min\{\sup(A), \sup(B)\}.$$
$$\sup\{a+b : a \in A, b \in B\} = \sup(A) + \sup(B).$$
$$\inf\{a+b : a \in A, b \in B\} = \inf(A) + \inf(B).$$
$$\sup\{a.b : a \in A, b \in B\} = \sup(A).\sup(B).$$
$$\inf\{a.b : a \in A, b \in B\} = \inf(A).\inf(B).$$

(2) Find all functions $h : \mathbb{R} \to \mathbb{R}$, satisfying $h(x+y) = h(x) + h(y)$ and $h(x.y) = h(x).h(y)$ for all x, y in $\mathbb{R}$. (Hint: You may need order properties and completeness of $\mathbb{R}$.)

(3) Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers converging to a real number x. Define a sequence $\{b_n\}_{n\in\mathbb{N}}$ by $b_n = a_{\sigma(n)}$. Show that $\{b_n\}$ converges to x. (Here $\{b_n\}$ is called a 'permutation' of $\{a_n\}$.)

(4) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. For $n \in \mathbb{N}$, take

$$c_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

These are known as Cesaro means of the sequence. Show that if $\{a_n\}$ is convergent so is $\{c_n\}$. Give an example to show that the converse is not true.

(5) Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be Cauchy sequences of real numbers. Without using the fact that they are convergent, show that $\{x_n + y_n\}_{n\in\mathbb{N}}$ and $\{x_n y_n\}_{n\in\mathbb{N}}$ are Cauchy.

(6) Prove that the sequence $\{v_n\}_{n\geq 1}$ defined recursively by $v_1 = 1$ and $v_{n+1} = v_n + \frac{1}{v_n}$, for $n \geq 1$, is not bounded.

(7) Show that the sequence $\{t^n\}_{n\in\mathbb{N}}$ converges if $t \in (-1, 1]$, properly diverges to $+\infty$ if $t > 1$ and diverges (not properly) for $t \in (-\infty, -1]$.

(8) Let L be the set of limit points of a bounded sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers. Suppose that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers in L. Show that if $\{y_n\}$ converges to y then $y \in L$.

(9) Compute limsup and liminf of following sequences:
   (i) $s_n = 5 + (-1)^n(2 + \frac{3}{2n})$ for $n \in \mathbb{N}$.
   (ii) $t_n = \frac{5}{(-2)^n} + \frac{2n}{4n^2-5}$ for $n \in \mathbb{N}$.

P.T.O.

(10) Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are bounded sequences of real numbers.

(i) Give an example, where

$$\limsup_{n\to\infty}(a_n + b_n) \neq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(ii) Show that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

and

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

Challenge Problem 2: [Answering this is optional. If you solve this problem, you need not do rest of the Assignment.] Suppose n is a natural number and $n \geq 2$. Show that there exist three natural numbers a, b and c (not necessarily distinct) such that

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

(Example: $\frac{4}{6} = \frac{1}{3} + \frac{1}{6} + \frac{1}{6}$.)