Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis -I

Home Assignment III — Due Date : 19 Dec 2021
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Notation: In the following when intervals $[a, b]$ are considered it is assumed that $a, b \in \mathbb{R}$ and $a < b$.

(1) Let $A \subseteq \mathbb{R}$ and let $c \in A$. Suppose $u : A \to \mathbb{R}$, $v : A \to \mathbb{R}$ are functions continuous at c. Define $w : A \to \mathbb{R}$ and $z : A \to \mathbb{R}$ by

$$w(t) = \max\{u(t), v(t)\},\ z(t) = \min\{u(t), v(t)\},\ t \in A.$$

Show that w, z are continuous at c. Hint: For any two real numbers a, b,

$$\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$$

and

$$\min\{a, b\} = \frac{1}{2}(a + b - |a - b|).$$

(2) **Definition:** Let I be an interval in $\mathbb{R}$. A function $g : I \to \mathbb{R}$ is said to be a convex function if

$$g(px + (1 - p)y) \leq pg(x) + (1 - p)g(y),\ 0 \leq p \leq 1, x, y \in I.$$

Pictorially this means that the graph of g between x and y stays below the line joining $(x, g(x))$ and $(y, g(y))$.

(i) Show that $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2,\ \forall x \in \mathbb{R}$ is convex.

(ii) Show that if $g : [a, b] \to \mathbb{R}$ is a convex function then for $a \leq s < t < u \leq b$,

$$\frac{g(t) - g(s)}{t - s} \leq \frac{g(u) - g(s)}{u - s} \leq \frac{g(u) - g(t)}{u - t}.$$

(iii) Show that if $g : [a, b] \to \mathbb{R}$ is convex function then g is continuous at every $c \in (a, b)$. However, g may not be continuous at a or b.

(3) (i) Show that $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^3,\ x \in \mathbb{R}$ is not uniformly continuous.
(ii) Show that the function $m : \mathbb{R} \to \mathbb{R}$ defined by $m(x) = \frac{5}{(1+x^2)},\ x \in \mathbb{R}$ is uniformly continuous.

(4) Let $g : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists $t \in [0, 1]$ such that $g(t) = t$. (Such points are known as fixed points of g). (Hint: Consider the function $h(t) = g(t) - t,\ t \in [0, 1]$ and use intermediate value theorem.)

(5) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Define $s : [a, b] \to \mathbb{R}$ by

$$s(x) = \sup\{f(t) : a \leq t \leq x\}, x \in [a, b].$$

Show that s is a continuous function.

(6) Let $g : [a, b] \to \mathbb{R}$ be a function. The set of discontinuity points of g is the set D defined by:

$$D = \{d \in [a, b] : g \text{ is discontinuous at } d\}.$$

Show that if g is a monotonic function then g is a countable set. (Hint: Consider rational numbers r satisfying $\lim_{x\to d-} g(x) < r < \lim_{x\to d+} g(x)$.)

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that there exists $d \in \mathbb{R}$ such that $f(x) = dx$ for all $x \in \mathbb{R}$.

(8) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(x) = h(5x)$ for all $x \in \mathbb{R}$. Show that h is a constant function.

(9) Show that there is no continuous function u on $\mathbb{R}$ such that $u(x)$ is irrational whenever x is rational and $u(x)$ is rational whenever x is irrational.

(10) Let B be a nonempty subset of $\mathbb{R}$. Define a function $k : \mathbb{R} \to \mathbb{R}$ by

$$k(x) = \inf\{|x - b| : b \in B\}.$$

(We may call $k(x)$ as the distance of x from B.) Show that k is a continuous function.

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