Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis-I

Home Assignment IV — Due Date: 24 Dec. 2021

1. (i) Express the repeating decimals $12.202120212021\ldots$ as the ratio of two integers.
   (ii) You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance $\frac{h}{\sqrt{2}}$. Find the total distance the ball travels up and down.
2. Prove or disprove the following:
   (i) If $\sum_{n=1}^{\infty} a_n$ is a series such that its sequence of partial sums $\{s_n\}_{n\in\mathbb{N}}$ satisfies
   $$\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \frac{1}{\sqrt{2}},$$
   then it is convergent.
   (ii) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then so is $\sum_{n=1}^{\infty} a_n^3$.
   (iii) The series $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}++-\cdots$ is convergent.
3. Show that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative reals such that $\{a_n\}_{n\in\mathbb{N}}$ is decreasing, then $\lim\limits_{n\to\infty} na_n = 0$.
4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive reals such that $t_n = \sum_{k=n}^{\infty} a_k$ for each $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{t_n}}$ is convergent.
5. Find the sum of the following series.
   (i) $\sum_{n=1}^{\infty} \frac{n}{n^4+n^2+1}$
   (ii) $\sum_{n=1}^{\infty} \left(\frac{3}{n^2+7n+12} + 3^{2+n}2^{1-3n}\right)$
6. Test the convergence of the following series.
   (i) $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$
   (ii) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{(2\cdot 4\cdot\cdots\cdot(2n))(3^n+1)}$
   (iii) $\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$
   (iv) $\sum_{n=1}^{\infty} a_n$ with $a_1=\sqrt{3}$ and $a_{n+1} = \frac{n}{n+1}a_n$ for all $n\in\mathbb{N}$.
7. Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$.
8. Prove that $\sum_{n=1}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2}$ for $|r|<1$.
9. Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be convergent with sums a and b, respectively. Show that if their Cauchy product $\sum_{n=0}^{\infty} c_n$ converges to c, then $c = ab$.
10. Prove that the Root test is 'stronger' than the Ratio test. More precisely, prove the following.
    (i) If we are able to use the Ratio Test for a series $\sum_{n=1}^{\infty} a_n$, where $a_n > 0$ for all $n\in\mathbb{N}$, then the Root Test works as well.
    (ii) There exists a series $\sum_{n=1}^{\infty} a_n$ with $a_n>0$ for all $n\in\mathbb{N}$ such that the Root Test indicates whether the series converges or diverges but the Ratio Test is inconclusive.

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