Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis -I

Home Assignment V — Due Date : 29 Dec 2021
Instructor: B V Rajarama Bhat

Notation: In the following when intervals $[a, b]$ are considered it is assumed that $a, b \in \mathbb{R}$ and $a < b$.

(1) Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = c$ has exactly two solutions for every $c \in \mathbb{R}$.

(2) Let $g : (1, 2) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to 1+} g(x) = 0$ and $\lim_{x\to 2-} g(x) = 5$. Show that there exists $x_0 \in (1, 2)$ such that $g(x_0) = 1 + \sqrt{3}$.

(3) Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$|h(x) - h(y)| \leq R|x - y|^2, \ \forall x, y \in \mathbb{R}.$$

Show that h must be a constant function.

(4) Let $f : (a, b) \to \mathbb{R}$ be a continuous function, differentiable at all points except possibly at $c \in (a, b)$. Suppose $\lim_{x\to c} f'(x) = L$ for some $L \in \mathbb{R}$, show that f is then differentiable at c and $f'(c) = L$.

(5) Suppose $g : [a, \infty) \to \mathbb{R}$ is a differentiable function and $g'(x) \geq 0$ for all $x \in (a, \infty)$. Show that $g'(a) \geq 0$.

(6) Suppose $a_1, a_2, \ldots, a_n$ are n-real numbers. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \sum_{j=1}^{n} (x - a_j)^2.$$

Find the unique global minimum point for g.

(7) Find points of relative extrema for the following functions. Identify the intervals in which it is increasing and the intervals where it is decreasing.
   (i) $g_1(x) = |(x - 2)(x - 3)|, x \in \mathbb{R}$.
   (ii) $g_2(x) = (|x| - 2)(|x| - 3), x \in \mathbb{R}$.

(8) Give an example to show that an uniformly continuous function need not be differentiable. Give another example to show that a differentiable function need not be uniformly continuous.

(9) Find first four terms in the Taylor expansion of the function $g(x) = \frac{1}{x}$ around the point $x_0 = 1$.

(10) Find first three terms of the Taylor expansion of the function $g : [-2, 2] \to \mathbb{R}$ defined by

$$g(x) = \frac{2x}{(x - 3)(x + 5)},$$

around the points $x_0 = 0$ and $x_1 = 1$.