ANALYSIS -I

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Lecture 1: Introduction

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- Mathematical as well as non-mathematical.
- Tell me which result you like most!

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My take

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- We do logical thinking.
- Given a set of statements, what are the statements we can deduce is what bothers us most of the time.
- We learn to make these deductions systematically.
- The statements we start with or which we take for granted are axioms.
- We think of some deductions as important or beautiful. We call them as theorems.

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- There is no contradiction here! Non-smoking also may cause cancer!
- Starting with a small set of axioms, the whole edifice of mathematics is built using logical deductions.

Applications

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- So on.
- We see structural, logical similarities in many different contexts.

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- We are living in a digital world. We convert all the information into digits. A sequence of 0's and 1's, The information could be audio, image, video, currency,...
- Keeping the information safe is done using cryptology. That also uses mathematics in a non-trivial way.

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- The setting should be clear. The statements should be clear, the deductions should be clear and so on.

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- An engineer, a physicist, and a mathematician were on a train heading north, and had just crossed the border into Scotland.
- The engineer looked out of the window and said "Look! Scottish sheep are black!"
- The physicist said, "No, no. Some Scottish sheep are black."
- The mathematician looked irritated and said: "All we can say is that there is one field, containing at least one sheep, of which at least one side is black, as of now."

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- Like for instance you may considering modeling COVID situation as applied mathematics.
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- It is very important to understand that these are broad classifications. There are no strict borders. More importantly very often methods and results become useful in another area. For instance, complex analysis is routinely used to do number theory.
- In other words all these topics are deeply inter-connected. Simply said, mathematics is one subject.
- You should learn basics of all the areas for now. Specialization comes only at an advanced level. You should not bother about it for now. Just have an open mind about all the areas.

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- In other words, there exists an element j which is contained in at least half the sets in $\mathcal{F}$.

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- $\mathcal{F}_3 = \{A \subseteq S : 1 \notin A\}$.
- $\mathcal{F}_4 = \{A \subseteq S : \sharp A = 2\}$.
- Then $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ satisfy conditions (i), (ii). $\mathcal{F}_4$ does not satisfy condition (iii).

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- $\mathcal{F}_1 = \{\emptyset, \{1,2\}, \{2,3\}, \{1,2,3\}\}$.
- $\sharp\mathcal{F}_1 = 4$; and we can take $j = 2$. There are three sets in $\mathcal{F}_1$ containing j.

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- $\mathcal{F}_2 = \{A \subseteq S : 1 \in A\}$.
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- END OF LECTURE 1.

Lecture 2: Set theory and Russell's paradox

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- $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$-the set of integers.

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- The main point here is that given an object we should be clear as to whether it is an element of the set or not.
- This is a requirement so that we do not have any confusion. Still the definition is only an informal one.

Russell's barber paradox

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- Let us see some more paradoxes of similar type.

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- What about the adjective 'HETEROLOGICAL? We again face a problem.

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- Is A ordinary or extraordinary? Either way we have a problem!

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- In the usual picture of graphs of functions on real line this is known as vertical line test. A graph of a function can not be touching a vertical line at more than one point.

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- Sometimes people call B, the co-domain as range of f. It is better to avoid that kind of terminology as it can lead to confusion.

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- It is fine, if some rooms are vacant. In other words, there could be $y \in B$ such that $y \neq f(x)$ for any $x \in A$.
- It is also fine if students are asked to share rooms. In other words it is possible to have x, x' in A, such that $f(x) = f(x')$.

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- While allotting rooms to students, injectivity or one-to-one means there is no sharing of rooms.

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- In the problem of allotting rooms to students it means that the hostel is full. That is all the rooms have got allotted.

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- Define $f_2 : \mathbb{Z} \to \mathbb{Z}$ by $f_2(n) = -n, \quad \forall n \in \mathbb{Z}$. Then f_2 is a bijection.
- Define $f_3 : \mathbb{Z} \to \mathbb{Z}$ by $f_3(n) = n^2$. Then f_3 is neither injective nor surjective.

Compositions of functions

- Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$ be functions. Then a new function $g \circ f : A \to C$ is got by taking
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- The out put of machine f is taken as input for g.

Inverse map

- Let A, B be non-empty sets and let $f : A \to B$ be a bijection. Then we see that for every $b \in B$ there exists unique $a \in A$ such that $f(a) = b$. Then we call a as $f^{-1}(b)$.

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- The identity map is a completely lazy machine where the output is same as the input.

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One sided inverses

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- Define $f : A \to B$ by $f(x) = 4$ and $f(y) = 6$.
- Define $g : B \to A$ by $g(4) = g(5) = x$ and $g(6) = y$.
- Then $g \circ f(x) = x$ and $g \circ f(y) = y$.
- So $g \circ f$ is the identity map on A. However, $f \circ g$ is not the identity map on B.

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- Proof: Take $h = g \circ f$. Suppose $f(a_1) = f(a_2)$ for some a_1, a_2 in A. Then by the definition of a function, $g(f(a_1)) = g(f(a_2))$. In other words, $h(a_1) = h(a_2)$. But h is assumed to be one to one. Hence $a_1 = a_2$. This shows that f is one to one.

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- Proof: Exercise!

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- Similarly $f^3(a) = (f \circ f \circ f)(a) = f(f(f(a)))$.
- More generally, we can define f^n for any natural number n.
- Note that in general you can not define f^2 when f is a function from one set to a different set.

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- Consider $h : \mathbb{Z} \to \mathbb{Z}$ defined by

$$h(n) = \begin{cases} 3k & \text{if} \quad n = 2k, \quad k \in \mathbb{Z} \\ 3k+1 & \text{if} \quad n = 4k+1 \quad k \in \mathbb{Z} \\ 3k-1 & \text{if} \quad n = 4k-1 \quad k \in \mathbb{Z} \end{cases}$$

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- END OF LECTURE 3.

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- Let us look at a few basic properties of the set of natural numbers and its subsets.

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- Note that clearly the minimal element of R is unique, for if both k, l are minimal then we have $k \leq l$ and $l \leq k$, and this means $k = l$.
- We also note that if $n \in R$, then the minimal element of R is contained in $\{1, 2, \ldots, n\} \bigcap R$. So the existence of minimum here is essentially a statement about finite sets.

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- Note. Here after we take it for granted that $\mathbb{N}$ has all these three properties.

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- Suppose we have a property P defined for natural numbers, where (i) 1 satisfies property P; (ii) If $m \in \mathbb{N}$ satisfies property P then $(m+1)$ satisfies property P. Then property P is satisfied by all natural numbers.

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- Hence $m+1 \in S$. Then by the principle of mathematical induction $S = \mathbb{N}$. In other words every natural number satisfies P.

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- Now put the ball you have in hand in bag and remove some other. Clearly the ball you have removed must be black color. Consider the balls in the bag. Now there are only m of them, also have to be of same color, same as the one ball we removed.

A fake theorem

- "Theorem": If you take bag full of balls all of them would have same color.
- "Proof":" We will prove this by induction.
- Let n be the number of balls in the bag.
- If $n = 1$, the claim is obvious. There is nothing to prove.
- Now assume the result for $n = m$ and we will prove it for $n = m + 1$.
- Suppose the bag has $m + 1$ balls. Remove one ball.
- Now there are m balls in the bag, and all of them have the same color, say black, by the induction hypothesis.
- Now put the ball you have in hand in bag and remove some other. Clearly the ball you have removed must be black color. Consider the balls in the bag. Now there are only m of them, also have to be of same color, same as the one ball we removed.
- So all the $m + 1$ balls are black. Quite Easily Done!

Pigeonhole principle

- Pigeonhole principle: Let m, n be natural numbers and $m < n$. Let
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- END OF LECTURE 4.

Lecture 5: Countable and Uncountable sets

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- We write $A \sim B$ if B is equipotent with A.

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- This completes the proof that equipotency ($\sim$) is an equivalence relation.

Finite and infinite sets

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- Example 5.4: $A = \{a, b, c\}$ and $B = \{x, y, z\}$ have same number of elements, namely 3, as both of them are equipotent with $\{1, 2, 3\}$.
- Even for infinite sets A, B we may informally say that A and B have same number of elements to mean that A and B are equipotent, even though we have not defined number of elements for infinite sets.

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- Definition 5.6: A set A is said to be countable if it is equipotent with $\mathbb{N}$ or if it is finite. It is said to be countably infinite if is countable and not finite. A set A is said to be uncountable if it is not countable.

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- Then new guest h_n can go to room number number $(2n - 1)$ and we are done.

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- You may verify that h is a bijection.

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- Moral of the story: For infinite sets, a subset may have as many elements as the full set.

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- In other words for infinite sets disjoint union of sets of equal number of elements may again have same number of elements.

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- This way we are able to exhaust all the elements of $\mathbb{N} \times \mathbb{N}$, without repeating any element twice.

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(2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
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$$

- Zig-zag counting.
- We count the elements here as $(1,1),(2,1),(1,2),(1,3),(2,2),(3,1),(4,1),(3,2),(2,3),(1,4),\ldots,$
- This way we are able to exhaust all the elements of $\mathbb{N} \times \mathbb{N}$, without repeating any element twice.
- In other words we have a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. In particular, $\mathbb{N} \times \mathbb{N}$ is countable.

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- Show that h is a bijection.
- Challenge Problem 3: Obtain another 'explicit' bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ different from $g, h, \tilde{g}, \tilde{h}$, where $\tilde{g}(m, n) = g(n, m)$, and $\tilde{h}(m, n) = h(n, m), \quad \forall m, n \in \mathbb{N} \times \mathbb{N}$.

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- This problem is not very clearly stated. But we leave it at that.

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- We saw that $\mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N}$ are all countable.
- Now it is time to see some uncountable sets.

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- Proof: Suppose that there exists a bijection $f : \mathbb{N} \to \mathbb{B}$. In particular f is a surjection.
- Then for every $i \in \mathbb{N}$, $f(i)$ is a binary sequence.

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- Look at the infinite matrix:

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- Form a binary sequence using the diagonal entries: $(w_{11}, w_{22}, w_{33}, \ldots)$.
- We flip the entries to get a new binary sequence, $v = (v_1, v_2, v_3, \ldots)$ where $v_j = 1 - w_{jj}$ for every $j \in \mathbb{N}$. Now we claim that v is not in the range of f.

Proof Continued

- $v \neq f(1)$ as $v = (v_1, v_2, \ldots)$, $f(1) = (w_{11}, w_{12}, \ldots)$ and $v_1 = 1 - w_{11} \neq w_{11}$. So the first entry does not match.

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- Clearly D is a subset of A, and hence it is an element of $P(A)$.
- We claim that D is not in the range of f. That would show that f is not surjective.

Proof continued

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- Now either $a_0 \in D$ or $a_0 \notin D$.

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- Therefore our assumption that D is in the range of f must be wrong. Consequently f is not surjective.

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- In other words, $c(j) := c_j$, is just the 'indicator function' of the set C.
- Now go back and see that the proof of last theorem and that of uncountability of $\mathbb{B}$ use the same idea!

Bigger and bigger infinities

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- Observe that $B^{\mathbb{N}}$ is same as the space of sequences with elements from B. In particular, if $B = \{0, 1\}$, then $B^{\mathbb{N}}$ is same as the space of binary sequences.

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- END OF LECTURE 6

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- If you wish, you may see the construction of real numbers in due course once you are fully familiar with various properties of real numbers.

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- Note that we have explicitly assumed that $1 \neq 0$

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- This axiom binds addition and multiplication.

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- Theorem 7.1 : (i) (Uniqueness of 0). If $e \in \mathbb{R}$ satisfies $a + e = e + a = a$ for all $a \in \mathbb{R}$, then $e = 0$. (ii) (uniqueness of 1). If $f \in \mathbb{R}$ satisfies $a.f = f.a = a$ for all $a \in \mathbb{R}$, then $f = 1$.

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- (iii) Given $a, b \in \mathbb{R}$ and $a.b = 0$.
- Now suppose $a \neq 0$, then a^{-1} exists and we get

$$a^{-1}.(a.b) = a^{-1}.0 = 0.$$

Hence by associativity of multiplication, $(a^{-1}.a).b = 0$, or $1.b = 0$, which implies $b = 0$. So either $a = 0$ or $b = 0$.

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- More generally, $n \in \mathbb{N}$ is identified with $1 + 1 + \cdots + 1(n \ \text{times})$.
- You may verify that all natural numbers are distinct.

Integers, rational numbers and irrational numbers

- $\mathbb{Z}$ is also thought of as a subset of $\mathbb{R}$: $0 \in \mathbb{Z}$ is identified with 0 of $\mathbb{R}$ and $-n$ for $n \in \mathbb{N}$ is just the additive inverse of n.

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- END OF LECTURE 7.

Lecture 8: Real Numbers : Order axioms

- We are assuming that there is a set called set of real numbers $\mathbb{R}$ with two binary operations', $+$, . , satisfying certain axioms.

Axioms for addition

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 -Existence of zero.
- A4. For every $a \in \mathbb{R}$, there exists an element '$-a$' in $\mathbb{R}$ such that

$$a + (-a) = (-a) + a = 0.$$

 -Existence of additive inverse. $-a$ is known as additive inverse of a.

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- M4. For every $a \in \mathbb{R}$, with $a \neq 0$, there exists an element 'a^{-1}' in $\mathbb{R}$ such that
$$a.a^{-1} = a^{-1}.a = 1.$$
-Existence of multiplicative inverse. a^{-1} is known as multiplicative inverse of a.

Distributivity

- D. For a, b, c in $\mathbb{R}$,

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- These axioms are known as algebraic axioms. They determine the 'algebraic structure' of real numbers.

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  (i) $a \in \mathbb{P}$;
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  [This is known as trichotomy property for real numbers.]
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- Warning: The notation $\mathbb{P}$ for positive real numbers is not standard. You may see $\mathbb{R}^+$, $(0, \infty)$ as some of the alternative notations for the set of positive real numbers.

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- Consider the set S of all natural numbers which are positive. Then $1 \in S$ and if $n \in S$, then $n + 1 \in S$.
- Now a simple application of mathematical induction shows that $n \in \mathbb{P}$ for every $n \in \mathbb{N}$.

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- Notation: For real numbers, a, b, we write $a < b$ or $b > a$ if $b - a \in \mathbb{P}$. We write $a \leq b$ or $b \geq a$ if $b - a \in \mathbb{P} \bigcup \{0\}$.

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- We may call a real number a as negative if $-a$ is positive.

Simple inequalities

- Theorem 8.2: Suppose a, b, c, d are real numbers. Then
  (i) If $a < b$, then $a + c < b + c$.
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  (iii) If $a < b$ and $c < d$, then $a + c < b + d$.
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- Often we show two real numbers a, b are equal by showing $a \leq b$ and $b \leq a$. The equality follows by trichotomy property.

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- Conversely, suppose $a^2 < b^2$. Hence $(b^2 - a^2) = (b + a)(b - a)$ is positive. As a, b are assumed to be positive, $(b + a)$ is positive. Now from Theorem 8.1 it is clear that for the product $(b + a)(b - a)$ to be positive, we also need $(b - a)$ positive.

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$|a + b| = |a - |b|| = a - |b| \leq a = |a| \leq |a| + |b|$.
- Similarly if a is positive and b is negative with $0 < a \leq |b|$, we get $|a + b| = |a - |b|| = |b| - a \leq |b| \leq |a| + |b|$. Other cases

Why is this triangle inequality?

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- Now it should be clear as to why this is called triangle inequality.
- You will see that this notion of distance has far reaching applications in Analysis.

No smallest or largest positive elements

- Theorem 8.5: (i) The set $\mathbb{P}$ has no least element, that is, there exists no positive real number α, such that $\alpha \leq a$ for every positive real number a. (ii) The set $\mathbb{P}$ has no largest element, that is, there exists no positive real number β, such that $a \leq \beta$ for every positive real number a.

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- END OF LECTURE 8.

Lecture 9: Real Numbers : Completeness Axiom

- We are assuming that there is a set called set of real numbers $\mathbb{R}$ with two binary operations', $+$, . , satisfying certain axioms.

Axioms for addition

- A1.

$$a + b = b + a, \quad \forall a, b \in \mathbb{R}.$$

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- A3. There exists an element called 'zero', denoted by '0' in $\mathbb{R}$ such that
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- A4. For every $a \in \mathbb{R}$, there exists an element '$-a$' in $\mathbb{R}$ such that
$$a + (-a) = (-a) + a = 0.$$
-Existence of additive inverse. $-a$ is known as additive inverse of a.

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Distributivity

- D. For a, b, c in $\mathbb{R}$,

$$a.(b + c) = a.b + a.c$$

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- These axioms are known as algebraic axioms. They determine the 'algebraic structure' of real numbers.

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- O3. If $a \in \mathbb{R}$, then exactly one of the following three properties is true:
  (i) $a \in \mathbb{P}$;
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  [This is known as trichotomy property for real numbers.]
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- In particular, $a > 0$ iff $a \in \mathbb{P}$. Similarly $a \geq 0$ iff $a \in \mathbb{P} \bigcup \{0\}$.

Boundedness

- Definition 9.1: A non-empty subset S of $\mathbb{R}$ is said to be bounded above if there exists $u \in \mathbb{R}$ such that

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Examples

- Example 9.4: Consider the set $S = \{1, 2, 3\}$. Then 4 is an upper bound for S. 5 is also an upper bound for S. -1 is a lower bound for S. $\frac{1}{2}$ is also a lower bound for S. Since S admits both lower and upper bounds, it is a bounded subset of $\mathbb{R}$.

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- Example 9.6: It is easily seen that $\mathbb{R}$ is neither bounded below nor bounded above.

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- Proof: Exercise.

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- Example 9.9: Suppose

$$S_1 = \{x \in \mathbb{R} : x \leq 1\};$$

$$S_2 = \{x \in \mathbb{R} : x < 1\}.$$

It is clear that 1 is the least upper bound for both S_1 and S_2. In particular, if u_0 is a least upper bound for S, then u_0 may or may not be in S.

Greatest lower bound

- Definition 9.10: Let S be a non-empty subset of $\mathbb{R}$, which is bounded below. Then $v_0 \in \mathbb{R}$ is said to be a greatest lower bound (or infimum) of S if

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It is clear that 1 is the greatest lower bound for both T_1 and T_2. In particular, if v_0 is a greatest lower bound for S, then v_0 may or may not be in S.

Equivalence

- Proposition 9.12: Let S be a non-empty subset of $\mathbb{R}$. Then the following are equivalent:
  (a) S is bounded above and $u_0 \in \mathbb{R}$ is the least upper bound of S.
  (b) $-S$ is bounded below and $-u_0 \in \mathbb{R}$ is the greatest lower bound of $-S$.

Completeness axiom of $\mathbb{R}$.

- C. Completeness axiom (Least upper bound property): Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

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- Proposition 9.13: Every non-empty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.
- Proof: Suppose $T \subset \mathbb{R}$ is non-empty and is bounded below. Then by consider $-T$ which is bounded above and appeal to the completeness axiom. If u_0 is the least upper bound of $-T$, we know that $-u_0$ is the greatest lower bound of T.

Notation

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$$\sup(S) = \begin{cases} \text{Least upper bound of } S & \text{if S is bounded above;} \\ \infty & \text{otherwise.} \end{cases}$$

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  $\sup(S) = -\inf(-S), \quad \inf(S) = -\sup(-S)$
- However, keep in mind that $-\infty, \infty$ are not real numbers.

A Characterization

- Theorem 9.14: Let S be a non-empty subset of $\mathbb{R}$ and let $u_0 \in \mathbb{R}$. Then $u_0 = \sup(S)$ if and only if
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- Proof: Suppose $u_0 = \sup(S)$. Consider any $\epsilon > 0$. Now if every $x \in S$ satisfies $x \leq u_0 - \epsilon$, then $u_0 - \epsilon$ is an upper bound for S. This contradicts the fact that u_0 is the least upper bound. Hence there exists some x_ϵ in S, such that $u_0 - \epsilon < x_\epsilon$.

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- Conversely suppose u_0 satisfies (i) and (ii). Now if u_0 is not the least upper bound of S, then there exists an upper bound u of S such that $u < u_0$. Take $\epsilon = u_0 - u$.
- As u is an upper bound of S, every $x \in S$ satisfies $x \leq u = u_0 - \epsilon$. This violates (ii). So u_0 must be the least upper bound of S.

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- Proof: Suppose $\mathbb{N}$ is bounded above.
- Then by the least upper bound property, $\mathbb{N}$ has a least upper bound, say u_0.

Continuation of proof

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- As x is a natural number $x + 1$ is also a natural number.
- Then $u_0 < x + 1$ is a contradiction, as u_0 is an upper bound for the set of natural numbers.
- Hence $\mathbb{N}$ can't be bounded above.

A corollary

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- Proof: Let $x \in \mathbb{R}$. If $n \leq x$ for every natural number n, then $\mathbb{N}$ is bounded above by x. Since $\mathbb{N}$ is not bounded above, there exists a natural number n such that $x < n$.

Archimedean property

- Theorem 9.17 (Archimedean property): Suppose $\epsilon \in \mathbb{R}$ and $\epsilon > 0$. Then given any $y \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that

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- That is, $\frac{y}{\epsilon} < n$ or $y < n\epsilon$.

Continuation

- Recall: Archimedean property: Suppose $\epsilon \in \mathbb{R}$ and $\epsilon > 0$. Then given any $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that

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- C- Completeness axiom.

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$$x \leq u, \quad \forall x \in S.$$

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- If S is non-empty and bounded above, its least upper bound is unique and is denoted by $\sup(S)$.

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- Now the result is a special case of Archimedean property with $x = 1$.

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- An integer $n \in \mathbb{Z}$ is said to be an odd number if it is not an even number. Odd integers are all of the form $2k + 1$ for some integer k, and conversely all integers of the form $2k + 1$ with $k \in \mathbb{Z}$ are all odd.

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- Proof. Exercise.

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- Theorem 10.2: There is no rational number x such that $x^2 = 2$.
- Proof: The proof is by contradiction.
- Suppose x is a rational number such that $x^2 = 2$.
- As x is a rational number, $x = \frac{p}{q}$, for some integers, p, q with $q \neq 0$.
- Without loss of generality, we may assume that p, q are relatively prime (they have no common factor bigger than 1). This is possible, because, if $p = rp_1$ and $q = rq_1$, with $r > 1$, we can write $x = \frac{p_1}{q_1}$.

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- Since squares of odd numbers are odd, p also must be even. Say, $p = 2k$, with $k \in \mathbb{Z}$.
- Then we get $4k^2 = 2q^2$ or $2k^2 = q^2$.

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- As $x^2 < 2^2$, we get $x < 2$. Therefore S is bounded above by 2.

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- It is easily seen that $-\sqrt{2}$ is the only other real number whose square 2.

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- $[x]$ is the unique integer satisfying $[x] \leq x < [x] + 1$.
- $x - [x]$ is known as the fractional part of x. Note that

$$0 \leq x - [x] < 1, \quad \forall x \in \mathbb{R}.$$

Intervals

- Notation: For any two real numbers a, b with $a < b$, we write

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- We call (a, b) as open interval and $[a, b]$ as closed interval. Intervals $[a, b)$ etc. are called semi-open intervals.

The density of rational and irrational numbers

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- Theorem 10.9: Suppose a, b are real numbers such that $a < b$.
  (i) Then there exists a rational number r such that $a < r < b$.
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- Theorem 10.9: Suppose a, b are real numbers such that $a < b$. (i) Then there exists a rational number r such that $a < r < b$. (ii) There exists an irrational number s such that $a < s < b$.
- Proof: (i) Case I: $a = 0$: We know that there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < b$. Since $\frac{1}{n}$ is rational, we are done.

Continuation

- Case II: $a > 0$. Now as $(b - a) > 0$, we can find $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < (b - a)$, or $1 < nb - na$, that is, $na + 1 < nb$.

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- Take $m = [na] + 1$. So $m \in \mathbb{N}$.

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- Case III: $a < 0$. The result for this case can be derived from Case I and Case II (Exercise).

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- END OF LECTURE 10.

Lecture 11: Real Numbers: Nested intervals property and Uncountability

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- This is only a visual aid for us. We are not connecting axioms of geometry with axioms of real line.

Nested Intervals

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- So intersection of a nested family of intervals can be empty.

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- Considering previous examples, the following theorem can be a bit of a surprise.

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- As $I_n \supseteq I_{n+1}$, we have $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ for every n.
- This means that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ for every n.

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- Combining the last two conclusions, we have

$$a_m \leq b_n, \quad \forall m \quad (ii)$$

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- Here if $u = v$, then $[u, v]$ is to be understood as the singleton $\{u\}$.

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- Let $\{x_1, x_2, \ldots\}$ be an enumeration of $[a, b]$. (This just means that $n \mapsto x_n$ is a bijective function from $\mathbb{N}$ to $[a, b]$.)

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- END OF LECTURE 11.

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- As every real number is its integer part plus the fractional part it suffices to consider real numbers in the interval $[0, 1)$ in binary and decimal systems.
- Qn: What is the difference between 1 and $0.9999999\cdots$?
- Ans: $1 = 0.999999\cdots$. In other words, they are equal.

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- This completes the proof by Mathematical Induction.

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- On the other hand if $b_1 = 1$, that is, $x \in [\frac{1}{2}, 1)$, the second binary digit b_2 is 0 if $x \in [\frac{1}{2}, \frac{3}{4})$ and $b_2 = 1$ if $x \in [\frac{3}{4}, 1)$.

Binary expansion: Continuation

- Continuing this way, if $b_1, b_2, \ldots, b_n$ are the first n-binary digits of x, then

$$\frac{b_1}{2} + \frac{b_2}{2^2} \cdots + \frac{b_n}{2^n} \leq x < \frac{b_1}{2^1} + \frac{b_2}{2^2} \cdots + \frac{(b_n + 1)}{2^n}.$$

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$$\frac{b_1}{2} + \frac{b_2}{2^2} \cdots + \frac{b_n}{2^n} \leq x < \frac{b_1}{2^1} + \frac{b_2}{2^2} \cdots + \frac{(b_n + 1)}{2^n}.$$

- In other words, taking

$$I_1 = [\frac{b_1}{2^1}, \quad \frac{(b_1 + 1)}{2^1}].$$

$$I_2 = [\frac{b_1}{2^1} + \frac{b_2}{2^2}, \quad \frac{b_1}{2^1} + \frac{(b_2 + 1)}{2^2}]$$

$$I_n = [\frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n}, \quad \frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{(b_n + 1)}{2^n}], \quad \forall n,$$

- we get a nested family of closed and bounded intervals:

$$I_1 \supset I_2 \supset I_2 \cdots$$

- satisfying $x \in I_n$ for every n.
- Hence $x \in \bigcap_{n \in \mathbb{N}} I_n$.

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- In other words, two different real numbers x, y would have different binary expansions.

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- In other words in $(0, 1)$, only numbers of the form $\frac{m}{2^k}$, with natural numbers m, k have two binary expansions.
- For instance, $\frac{1}{2}$ is expressed as $0.10000000\ldots$ using the first option and as $0.0111111111\ldots$ through the second option.

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- Similarly $1 = \sup\{\frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^n} : n \in \mathbb{N}\}$.

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- In such cases, we say that x has a terminating decimal expansion. (It ends either with a sequence of 0's or with a sequence of 9's.)

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  The sequence $d_1, d_2, \ldots$ is uniquely determined unless $x = \frac{m}{M^k}$ for some natural numbers m, k. Further, if $x = \frac{m}{M^k}$ then x has two possible expressions, one terminating with 0's and another terminating with $(M-1)$'s.

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- END OF LECTURE 12

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- Then by mathematical induction we have a sequence $\{x_1, x_2, \dots\}$ of distinct elements in S. Clearly $T = \{x_n : n \in \mathbb{N}\}$ is equipotent with $\mathbb{N}$.

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- Clearly $T \bigcup C$ is equipotent with T.
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$$(\tilde{f})(x) = \begin{cases} f(x) & x \in T; \\ x & x \in S \setminus T \end{cases}$$

is seen to be a bijection from S to $S \bigcup C$ and this completes the proof.

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$$(\tilde{f})(x) = \begin{cases} f(x) & x \in T; \\ x & x \in S \backslash T \end{cases}$$

  is seen to be a bijection from S to $S \bigcup C$ and this completes the proof.
- Corollary 13.4: If S is an uncountable set and $T \subset S$ is countable then S is equipotent with $S \backslash T$.

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- (v) It is an exercise to cover all the remaining cases.

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- Show that $\mathbb{R} \times \mathbb{R}$ is equipotent with $\mathbb{R}$. More generally, show that $\mathbb{R}^n$ is equipotent with $\mathbb{R}$ for any $n \in \mathbb{N}$.

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14. Direct and inverse images of functions

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- Note that for any element x of X, $f(\{x\}) = \{f(x)\}$, which is the singleton set containing $f(x)$ and is different from the element $f(x)$. This distinction between elements and singleton sets should always be maintained to avoid confusion.

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- Similarly, you can show $f(A) \bigcup f(B) \subseteq f(A \bigcup B)$ and conclude that $f(A \bigcup B) = f(A) \bigcup f(B)$.

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- Proof: (a) follows from the definition of surjectivity. (b) and (c) are interesting exercises.

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- For the example, $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = x^2, \ \ \forall x \in \mathbb{R}$, we see that $g^{-1}(\{0\}) = \{0\}$ and $g^{-1}([0, \infty)) = \mathbb{R}$.

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$$f^{-1}(f(A)) \supseteq A$$

  and the equality may not hold.
- (b) For any subset V of Y,

$$f(f^{-1}(V)) \subseteq V$$

  and the equality may not hold.

Continuation

- It is indeed amazing that the inverse image f^{-1} respects all set theoretic operations with no conditions imposed on f. This is a very useful fact to remember.
- The proof of Theorem 14.5 is also as an exercise.
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