ANALYSIS -I

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Lecture 1: Introduction

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- Mathematical as well as non-mathematical.
- Tell me which result you like most!

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- Answers received:

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My take

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- We do logical thinking.
- Given a set of statements, what are the statements we can deduce is what bothers us most of the time.
- We learn to make these deductions systematically.
- The statements we start with or which we take for granted are axioms.
- We think of some deductions as important or beautiful. We call them as theorems.

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- There is no contradiction here! Non-smoking also may cause cancer!
- Starting with a small set of axioms, the whole edifice of mathematics is built using logical deductions.

Applications

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- So on.
- We see structural, logical similarities in many different contexts.

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- Keeping the information safe is done using cryptology. That also uses mathematics in a non-trivial way.

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- The setting should be clear. The statements should be clear, the deductions should be clear and so on.

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- An engineer, a physicist, and a mathematician were on a train heading north, and had just crossed the border into Scotland.
- The engineer looked out of the window and said "Look! Scottish sheep are black!"
- The physicist said, "No, no. Some Scottish sheep are black."
- The mathematician looked irritated and said: "All we can say is that there is one field, containing at least one sheep, of which at least one side is black, as of now."

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- It is very important to understand that these are broad classifications. There are no strict borders. More importantly very often methods and results become useful in another area. For instance, complex analysis is routinely used to do number theory.
- In other words all these topics are deeply inter-connected. Simply said, mathematics is one subject.
- You should learn basics of all the areas for now. Specialization comes only at an advanced level. You should not bother about it for now. Just have an open mind about all the areas.

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- T. M. Apostol: Mathematical Analysis.

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- Show that there exists $j \in S$ such that

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- In other words, there exists an element j which is contained in at least half the sets in $\mathcal{F}$.

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- $\mathcal{F}_2 = \{A \subseteq S : 1 \in A\}$.
- $\mathcal{F}_3 = \{A \subseteq S : 1 \notin A\}$.
- $\mathcal{F}_4 = \{A \subseteq S : \sharp A = 2\}$.
- Then $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ satisfy conditions (i), (ii). $\mathcal{F}_4$ does not satisfy condition (iii).

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- $\mathcal{F}_1 = \{\emptyset, \{1,2\}, \{2,3\}, \{1,2,3\}\}$.
- $\sharp\mathcal{F}_1 = 4$; and we can take $j = 2$. There are three sets in $\mathcal{F}_1$ containing j.

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- $\mathcal{F}_2 = \{A \subseteq S : 1 \in A\}$.
- $\sharp\mathcal{F}_2 = 2^9$; and we can take $j = 1$ and $\sharp\{A \in \mathcal{F}_2 : j \in A\} = 2^9$.

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- $\mathcal{F}_3 = \{A \subseteq S : 1 \notin A\}$.
- $\sharp\mathcal{F}_1 = 2^9$; and we can take $j = 2$ (or any number in S different from 1) and $\sharp\{A \in \mathcal{F}_3 : j \in A\} = 2^8$.

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- END OF LECTURE 1.