ANALYSIS -I

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- C- Completeness axiom.

Recall: Completeness axiom

- Definition 9.1: A non-empty subset S of $\mathbb{R}$ is said to be bounded above if there exists $u \in \mathbb{R}$ such that

$$x \leq u, \quad \forall x \in S.$$

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- C. Completeness axiom (Least upper bound property): Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.
- If S is non-empty and bounded above, its least upper bound is unique and is denoted by $\sup(S)$.

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- Now the result is a special case of Archimedean property with $x = 1$.

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- Proof. Exercise.

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- Suppose x is a rational number such that $x^2 = 2$.
- As x is a rational number, $x = \frac{p}{q}$, for some integers, p, q with $q \neq 0$.
- Without loss of generality, we may assume that p, q are relatively prime (they have no common factor bigger than 1). This is possible, because, if $p = rp_1$ and $q = rq_1$, with $r > 1$, we can write $x = \frac{p_1}{q_1}$.

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- This completes the proof.

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- As $x^2 < 2^2$, we get $x < 2$. Therefore S is bounded above by 2.

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- We denote s, by $\sqrt{2}$.
- It is easily seen that $-\sqrt{2}$ is the only other real number whose square 2.

Other roots

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- $x - [x]$ is known as the fractional part of x. Note that

$$0 \leq x - [x] < 1, \quad \forall x \in \mathbb{R}.$$

Intervals

- Notation: For any two real numbers a, b with $a < b$, we write

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- We call (a, b) as open interval and $[a, b]$ as closed interval. Intervals $[a, b)$ etc. are called semi-open intervals.

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  (i) Then there exists a rational number r such that $a < r < b$.
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- Proof: (i) Case I: $a = 0$: We know that there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < b$. Since $\frac{1}{n}$ is rational, we are done.

Continuation

- Case II: $a > 0$. Now as $(b - a) > 0$, we can find $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < (b - a)$, or $1 < nb - na$, that is, $na + 1 < nb$.

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- Case III: $a < 0$. The result for this case can be derived from Case I and Case II (Exercise).

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- END OF LECTURE 10.