ANALYSIS -I

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Lecture 11: Real Numbers: Nested intervals property and Uncountability

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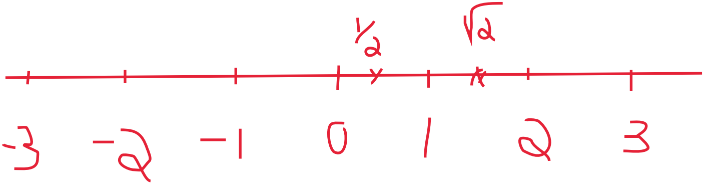

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- This is only a visual aid for us. We are not connecting axioms of geometry with axioms of real line.

Nested Intervals

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- Proof: Clearly $0 \in (-\frac{1}{n}, \frac{1}{n})$ for every $n \in \mathbb{N}$, and hence $0 \in \bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})$.

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- This completes the proof.

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- So intersection of a nested family of intervals can be empty.

One more example

- For $n \in \mathbb{N}$ take $K_n = [n, \infty) = \{x \in \mathbb{R} : n \leq x\}$.

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- Considering previous examples, the following theorem can be a bit of a surprise.

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- As $I_n \supseteq I_{n+1}$, we have $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ for every n.
- This means that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ for every n.

Continuation

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$$a_n \leq u, \qquad (i)$$

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  Hence $a_m \leq b_n$ for $1 \leq m \leq n$.
- For $m \geq n$, $I_m \subseteq I_n$, and hence $a_n \leq a_m < b_m \leq b_n$. In particular, $a_m \leq b_n$.

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- For $m \geq n$, $I_m \subseteq I_n$, and hence $a_n \leq a_m < b_m \leq b_n$. In particular, $a_m \leq b_n$.
- Combining the last two conclusions, we have

$$a_m \leq b_n, \quad \forall m \qquad (ii)$$

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- From (ii), b_n is an upper bound for A. Since u is the least upper bound, we get

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$$[u, v] \subseteq \bigcap_{n\in\mathbb{N}} I_n.$$
- Here if $u = v$, then $[u, v]$ is to be understood as the singleton $\{u\}$.

The Singleton

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- Let $\{x_1, x_2, \ldots\}$ be an enumeration of $[a, b]$. (This just means that $n \mapsto x_n$ is a bijective function from $\mathbb{N}$ to $[a, b]$.)

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- END OF LECTURE 11.