ANALYSIS -I

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 12: Real Numbers: Binary and Decimal systems

- Consider $\mathbb{R}$, the set of real numbers.

Lecture 12: Real Numbers: Binary and Decimal systems

- Consider $\mathbb{R}$, the set of real numbers.
- We want to look at the familiar binary and decimal systems of writing real numbers.

Lecture 12: Real Numbers: Binary and Decimal systems

- Consider $\mathbb{R}$, the set of real numbers.
- We want to look at the familiar binary and decimal systems of writing real numbers.
- Binary expansion for integers: We know that any natural number can be written uniquely as $c_n.2^n + c_{n-1}.2^{n-1} + \cdots + c_1.2 + c_0$, for some $n \in \mathbb{N}$ with each $c_j \in \{0, 1\}$ ($c_n \neq 0$).

Lecture 12: Real Numbers: Binary and Decimal systems

- Consider $\mathbb{R}$, the set of real numbers.
- We want to look at the familiar binary and decimal systems of writing real numbers.
- Binary expansion for integers: We know that any natural number can be written uniquely as $c_n.2^n + c_{n-1}.2^{n-1} + \cdots + c_1.2 + c_0$, for some $n \in \mathbb{N}$ with each $c_j \in \{0, 1\}$ ($c_n \neq 0$).
- As every real number is its integer part plus the fractional part it suffices to consider real numbers in the interval $[0, 1)$ in binary and decimal systems.

Lecture 12: Real Numbers: Binary and Decimal systems

- Consider $\mathbb{R}$, the set of real numbers.
- We want to look at the familiar binary and decimal systems of writing real numbers.
- Binary expansion for integers: We know that any natural number can be written uniquely as $c_n.2^n + c_{n-1}.2^{n-1} + \cdots + c_1.2 + c_0$, for some $n \in \mathbb{N}$ with each $c_j \in \{0, 1\}$ $(c_n \neq 0)$.
- As every real number is its integer part plus the fractional part it suffices to consider real numbers in the interval $[0, 1)$ in binary and decimal systems.
- Qn: What is the difference between 1 and $0.9999999\cdots$?

Lecture 12: Real Numbers: Binary and Decimal systems

- Consider $\mathbb{R}$, the set of real numbers.
- We want to look at the familiar binary and decimal systems of writing real numbers.
- Binary expansion for integers: We know that any natural number can be written uniquely as $c_n.2^n + c_{n-1}.2^{n-1} + \cdots + c_1.2 + c_0$, for some $n \in \mathbb{N}$ with each $c_j \in \{0, 1\}$ $(c_n \neq 0)$.
- As every real number is its integer part plus the fractional part it suffices to consider real numbers in the interval $[0, 1)$ in binary and decimal systems.
- Qn: What is the difference between 1 and $0.9999999\cdots$?
- Ans: $1 = 0.999999\cdots$. In other words, they are equal.

Bernoulli's inequality

- Theorem 12.1 (Bernoulli's inequality): If $x \in \mathbb{R}$ with $x > -1$, then

$$(1 + x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}.$$

Bernoulli's inequality

- Theorem 12.1 (Bernoulli's inequality): If $x \in \mathbb{R}$ with $x > -1$, then

$$(1+x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}.$$

- Proof: This we prove by induction on n.

Bernoulli's inequality

- Theorem 12.1 (Bernoulli's inequality): If $x \in \mathbb{R}$ with $x > -1$, then
$$(1 + x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}.$$
- Proof: This we prove by induction on n.
- For $n = 1$, clearly the equality holds.

Bernoulli's inequality

- Theorem 12.1 (Bernoulli's inequality): If $x \in \mathbb{R}$ with $x > -1$, then
$$(1+x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}.$$
- Proof: This we prove by induction on n.
- For $n = 1$, clearly the equality holds.
- Assume the result for $n = m$, so we have $(1+x)^m \geq 1 + mx$.

Bernoulli's inequality

- Theorem 12.1 (Bernoulli's inequality): If $x \in \mathbb{R}$ with $x > -1$, then
$$(1+x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}.$$
- Proof: This we prove by induction on n.
- For $n = 1$, clearly the equality holds.
- Assume the result for $n = m$, so we have $(1+x)^m \geq 1 + mx$.
- Note that as $x > -1$, $1 + x > 0$.

Bernoulli's inequality

- Theorem 12.1 (Bernoulli's inequality): If $x \in \mathbb{R}$ with $x > -1$, then

$$(1 + x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}.$$

- Proof: This we prove by induction on n.
- For $n = 1$, clearly the equality holds.
- Assume the result for $n = m$, so we have $(1 + x)^m \geq 1 + mx$.
- Note that as $x > -1$, $1 + x > 0$.
- Now using the induction hypothesis,

$$\begin{aligned} (1 + x)^{m+1} &= (1 + x)^m.(1 + x) \\ &\geq (1 + mx)(1 + x) \\ &= 1 + x + mx + mx^2 \\ &\geq 1 + (m + 1)x \end{aligned}$$

as $mx^2 \geq 0$.

Bernoulli's inequality

- Theorem 12.1 (Bernoulli's inequality): If $x \in \mathbb{R}$ with $x > -1$, then
$$(1+x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}.$$
- Proof: This we prove by induction on n.
- For $n = 1$, clearly the equality holds.
- Assume the result for $n = m$, so we have $(1+x)^m \geq 1 + mx$.
- Note that as $x > -1$, $1 + x > 0$.
- Now using the induction hypothesis,
$$\begin{aligned} (1+x)^{m+1} &= (1+x)^m.(1+x) \\ &\geq (1+mx)(1+x) \\ &= 1 + x + mx + mx^2 \\ &\geq 1 + (m+1)x \end{aligned}$$
as $mx^2 \geq 0$.
- Hence the inequality is true for $n = m + 1$.

Bernoulli's inequality

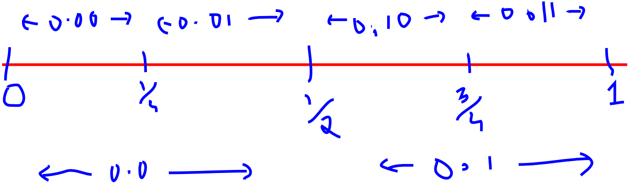
- Theorem 12.1 (Bernoulli's inequality): If $x \in \mathbb{R}$ with $x > -1$, then
$$(1+x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}.$$
- Proof: This we prove by induction on n.
- For $n = 1$, clearly the equality holds.
- Assume the result for $n = m$, so we have $(1+x)^m \geq 1 + mx$.
- Note that as $x > -1$, $1 + x > 0$.
- Now using the induction hypothesis,
$$\begin{aligned} (1+x)^{m+1} &= (1+x)^m.(1+x) \\ &\geq (1+mx)(1+x) \\ &= 1 + x + mx + mx^2 \\ &\geq 1 + (m+1)x \end{aligned}$$
as $mx^2 \geq 0$.
- Hence the inequality is true for $n = m + 1$.
- This completes the proof by Mathematical Induction.

Binary system

- We divide the interval $[0, 1)$ into two parts, $[0, 1/2)$ and $[\frac{1}{2}, 1)$, and then sub-divide them into two more equal pieces and so on.

Binary system

- We divide the interval $[0,1)$ into two parts, $[0,1/2)$ and $[\frac{1}{2},1)$, and then sub-divide them into two more equal pieces and so on.



Binary system

- We divide the interval $[0, 1)$ into two parts, $[0, 1/2)$ and $[\frac{1}{2}, 1)$, and then sub-divide them into two more equal pieces and so on.
- We have $[0, 1) = [0, \frac{1}{2}) \bigcup [\frac{1}{2}, 1)$
- If $x \in [0, \frac{1}{2})$, the first binary digit b_1 of x is 0. If $x \in [\frac{1}{2}, 1)$, the first binary digit b_1 of x is 1.

Binary system

- We divide the interval $[0, 1)$ into two parts, $[0, 1/2)$ and $[\frac{1}{2}, 1)$, and then sub-divide them into two more equal pieces and so on.
- We have $[0, 1) = [0, \frac{1}{2}) \bigcup [\frac{1}{2}, 1)$
- If $x \in [0, \frac{1}{2})$, the first binary digit b_1 of x is 0. If $x \in [\frac{1}{2}, 1)$, the first binary digit b_1 of x is 1.
- Here we have made a choice to put the mid-point with the right interval. We can opt to the mid-point with the left interval. This option we will explore later on.
- Consider the case where $b_1 = 0$. Now $x \in [0, \frac{1}{2})$. To determine the second digit, divide $[0, \frac{1}{2})$ into two parts.

Binary system

- We divide the interval $[0, 1)$ into two parts, $[0, 1/2)$ and $[\frac{1}{2}, 1)$, and then sub-divide them into two more equal pieces and so on.
- We have $[0, 1) = [0, \frac{1}{2}) \bigcup [\frac{1}{2}, 1)$
- If $x \in [0, \frac{1}{2})$, the first binary digit b_1 of x is 0. If $x \in [\frac{1}{2}, 1)$, the first binary digit b_1 of x is 1.
- Here we have made a choice to put the mid-point with the right interval. We can opt to the mid-point with the left interval. This option we will explore later on.
- Consider the case where $b_1 = 0$. Now $x \in [0, \frac{1}{2})$. To determine the second digit, divide $[0, \frac{1}{2})$ into two parts.
- If $x \in [0, \frac{1}{4})$, the second binary digit b_2 of x is 0. If $x \in [\frac{1}{4}, \frac{1}{2})$ the second binary digit b_2 of x is 1.

Binary system

- We divide the interval $[0,1)$ into two parts, $[0,1/2)$ and $[\frac{1}{2},1)$, and then sub-divide them into two more equal pieces and so on.
- We have $[0,1)=[0,\frac{1}{2})\bigcup[\frac{1}{2},1)$
- If $x\in[0,\frac{1}{2})$, the first binary digit b_1 of x is 0. If $x\in[\frac{1}{2},1)$, the first binary digit b_1 of x is 1.
- Here we have made a choice to put the mid-point with the right interval. We can opt to the mid-point with the left interval. This option we will explore later on.
- Consider the case where $b_1=0$. Now $x\in[0,\frac{1}{2})$. To determine the second digit, divide $[0,\frac{1}{2})$ into two parts.
- If $x\in[0,\frac{1}{4})$, the second binary digit b_2 of x is 0. If $x\in[\frac{1}{4},\frac{1}{2})$ the second binary digit b_2 of x is 1.
- On the other hand if $b_1=1$, that is, $x\in[\frac{1}{2},1)$, the second binary digit b_2 is 0 if $x\in[\frac{1}{2},\frac{3}{4})$ and $b_2=1$ if $x\in[\frac{3}{4},1)$.

Binary expansion: Continuation

- Continuing this way, if $b_1, b_2, \ldots, b_n$ are the first n-binary digits of x, then

$$\frac{b_1}{2} + \frac{b_2}{2^2} \cdots + \frac{b_n}{2^n} \leq x < \frac{b_1}{2^1} + \frac{b_2}{2^2} \cdots + \frac{(b_n + 1)}{2^n}.$$

Binary expansion: Continuation

- Continuing this way, if $b_1, b_2, \ldots, b_n$ are the first n-binary digits of x, then

$$\frac{b_1}{2} + \frac{b_2}{2^2} \cdots + \frac{b_n}{2^n} \leq x < \frac{b_1}{2^1} + \frac{b_2}{2^2} \cdots + \frac{(b_n+1)}{2^n}.$$

- In other words, taking

$$I_1 = [\frac{b_1}{2^1}, \quad \frac{(b_1+1)}{2^1}].$$

$$I_2 = [\frac{b_1}{2^1} + \frac{b_2}{2^2}, \quad \frac{b_1}{2^1} + \frac{(b_2+1)}{2^2}]$$

$$I_n = [\frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n}, \quad \frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{(b_n+1)}{2^n}], \quad \forall n,$$

Binary expansion: Continuation

- Continuing this way, if $b_1, b_2, \ldots, b_n$ are the first n-binary digits of x, then

$$\frac{b_1}{2} + \frac{b_2}{2^2} \cdots + \frac{b_n}{2^n} \leq x < \frac{b_1}{2^1} + \frac{b_2}{2^2} \cdots + \frac{(b_n + 1)}{2^n}.$$

- In other words, taking

$$I_1 = [\frac{b_1}{2^1}, \quad \frac{(b_1 + 1)}{2^1}].$$

$$I_2 = [\frac{b_1}{2^1} + \frac{b_2}{2^2}, \quad \frac{b_1}{2^1} + \frac{(b_2 + 1)}{2^2}]$$

$$I_n = [\frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n}, \quad \frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{(b_n + 1)}{2^n}], \quad \forall n,$$

- we get a nested family of closed and bounded intervals:

$$I_1 \supset I_2 \supset I_2 \cdots$$

Binary expansion: Continuation

- Continuing this way, if $b_1, b_2, \dots, b_n$ are the first n-binary digits of x, then

$$\frac{b_1}{2} + \frac{b_2}{2^2} \cdots + \frac{b_n}{2^n} \leq x < \frac{b_1}{2^1} + \frac{b_2}{2^2} \cdots + \frac{(b_n + 1)}{2^n}.$$

- In other words, taking

$$I_1 = [\frac{b_1}{2^1}, \quad \frac{(b_1 + 1)}{2^1}].$$

$$I_2 = [\frac{b_1}{2^1} + \frac{b_2}{2^2}, \quad \frac{b_1}{2^1} + \frac{(b_2 + 1)}{2^2}]$$

$$I_n = [\frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n}, \quad \frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{(b_n + 1)}{2^n}], \quad \forall n,$$

- we get a nested family of closed and bounded intervals:

$$I_1 \supset I_2 \supset I_2 \cdots$$

- satisfying $x \in I_n$ for every n.

Binary expansion: Continuation

- Continuing this way, if $b_1, b_2, \ldots, b_n$ are the first n-binary digits of x, then
$$\frac{b_1}{2} + \frac{b_2}{2^2} \cdots + \frac{b_n}{2^n} \leq x < \frac{b_1}{2^1} + \frac{b_2}{2^2} \cdots + \frac{(b_n + 1)}{2^n}.$$
- In other words, taking
$$I_1 = [\frac{b_1}{2^1}, \quad \frac{(b_1 + 1)}{2^1}].$$
$$I_2 = [\frac{b_1}{2^1} + \frac{b_2}{2^2}, \quad \frac{b_1}{2^1} + \frac{(b_2 + 1)}{2^2}]$$
$$I_n = [\frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n}, \quad \frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{(b_n + 1)}{2^n}], \quad \forall n,$$
- we get a nested family of closed and bounded intervals:
$$I_1 \supset I_2 \supset I_2 \cdots$$
- satisfying $x \in I_n$ for every n.
- Hence $x \in \bigcap_{n \in \mathbb{N}} I_n$.

Continuation

- By Bernoulli's inequality (taking $x = 1$) $2^n = (1+1)^n \geq 1 + n$.

Continuation

- By Bernoulli's inequality (taking $x = 1$) $2^n = (1+1)^n \geq 1+n$.
- In particular, for $\epsilon > 0$, there exists $n \in \mathbb{N}$, such that $0 < \frac{1}{2^n} < \frac{1}{n+1} < \epsilon$.

Continuation

- By Bernoulli's inequality (taking $x = 1$) $2^n = (1+1)^n \geq 1 + n$.
- In particular, for $\epsilon > 0$, there exists $n \in \mathbb{N}$, such that $0 < \frac{1}{2^n} < \frac{1}{n+1} < \epsilon$.
- Consequently, $\inf\{\frac{(b_n+1)}{2^n} - \frac{b_n}{2^n} : n \in \mathbb{N}\} = \inf\{\frac{1}{2^n} : n \in \mathbb{N}\} = 0$.

Continuation

- By Bernoulli's inequality (taking $x = 1$) $2^n = (1+1)^n \geq 1+n$.
- In particular, for $\epsilon > 0$, there exists $n \in \mathbb{N}$, such that $0 < \frac{1}{2^n} < \frac{1}{n+1} < \epsilon$.
- Consequently, $\inf\{\frac{(b_n+1)}{2^n} - \frac{b_n}{2^n} : n \in \mathbb{N}\} = \inf\{\frac{1}{2^n} : n \in \mathbb{N}\} = 0$.
- Then by Theorem 11.5, $\bigcap_{n\in\mathbb{N}} I_n$ is singleton.

Continuation

- By Bernoulli's inequality (taking $x = 1$) $2^n = (1+1)^n \geq 1 + n$.
- In particular, for $\epsilon > 0$, there exists $n \in \mathbb{N}$, such that $0 < \frac{1}{2^n} < \frac{1}{n+1} < \epsilon$.
- Consequently, $\inf\{\frac{(b_n+1)}{2^n} - \frac{b_n}{2^n} : n \in \mathbb{N}\} = \inf\{\frac{1}{2^n} : n \in \mathbb{N}\} = 0$.
- Then by Theorem 11.5, $\bigcap_{n\in\mathbb{N}} I_n$ is singleton.
- Hence $\bigcap_{n\in\mathbb{N}} I_n = \{x\}$.

Continuation

- By Bernoulli's inequality (taking $x = 1$) $2^n = (1+1)^n \geq 1+n$.
- In particular, for $\epsilon > 0$, there exists $n \in \mathbb{N}$, such that $0 < \frac{1}{2^n} < \frac{1}{n+1} < \epsilon$.
- Consequently, $\inf\{\frac{(b_n+1)}{2^n} - \frac{b_n}{2^n} : n \in \mathbb{N}\} = \inf\{\frac{1}{2^n} : n \in \mathbb{N}\} = 0$.
- Then by Theorem 11.5, $\bigcap_{n\in\mathbb{N}} I_n$ is singleton.
- Hence $\bigcap_{n\in\mathbb{N}} I_n = \{x\}$.
- This shows that the binary digits of x, determines x.

Continuation

- By Bernoulli's inequality (taking $x = 1$) $2^n = (1+1)^n \geq 1 + n$.
- In particular, for $\epsilon > 0$, there exists $n \in \mathbb{N}$, such that $0 < \frac{1}{2^n} < \frac{1}{n+1} < \epsilon$.
- Consequently, $\inf\{\frac{(b_n+1)}{2^n} - \frac{b_n}{2^n} : n \in \mathbb{N}\} = \inf\{\frac{1}{2^n} : n \in \mathbb{N}\} = 0$.
- Then by Theorem 11.5, $\bigcap_{n\in\mathbb{N}} I_n$ is singleton.
- Hence $\bigcap_{n\in\mathbb{N}} I_n = \{x\}$.
- This shows that the binary digits of x, determines x.
- In other words, two different real numbers x, y would have different binary expansions.

Second Option in Binary expansion

- Now we explore the Second Option.

Second Option in Binary expansion

- Now we explore the Second Option.
- We divide the interval $(0,1]$ as $(0,\frac{1}{2}]\bigcup(\frac{1}{2},1]$.

Second Option in Binary expansion

- Now we explore the Second Option.
- We divide the interval $(0,1]$ as $(0,\frac{1}{2}]\bigcup(\frac{1}{2},1]$.
- Then $(0,\frac{1}{2}]$ as $(0,\frac{1}{2^2}]\bigcup(\frac{1}{2^2},\frac{1}{2}]$ and $(\frac{1}{2},1]$ as $(\frac{1}{2},\frac{3}{2^2}]\bigcup(\frac{3}{2^2},1]$.

Second Option in Binary expansion

- Now we explore the Second Option.
- We divide the interval $(0,1]$ as $(0,\frac{1}{2}]\bigcup(\frac{1}{2},1]$.
- Then $(0,\frac{1}{2}]$ as $(0,\frac{1}{2^2}]\bigcup(\frac{1}{2^2},\frac{1}{2}]$ and $(\frac{1}{2},1]$ as $(\frac{1}{2},\frac{3}{2^2}]\bigcup(\frac{3}{2^2},1]$.
- This way we get a possibly new binary expansion, say the digits are $c_1, c_2, \ldots$, satisfying

$$\frac{c_1}{2^1}+\frac{c_2}{2^2}\cdots+\frac{c_n}{2^n}\leq x\leq\frac{c_1}{2^1}+\frac{c_2}{2^2}\cdots+\frac{c_n+1}{2^n}.$$

Second Option in Binary expansion

- Now we explore the Second Option.
- We divide the interval $(0,1]$ as $(0,\frac{1}{2}]\bigcup(\frac{1}{2},1]$.
- Then $(0,\frac{1}{2}]$ as $(0,\frac{1}{2^2}]\bigcup(\frac{1}{2^2},\frac{1}{2}]$ and $(\frac{1}{2},1]$ as $(\frac{1}{2},\frac{3}{2^2}]\bigcup(\frac{3}{2^2},1]$.
- This way we get a possibly new binary expansion, say the digits are $c_1, c_2, \ldots$, satisfying

$$\frac{c_1}{2^1}+\frac{c_2}{2^2}\cdots+\frac{c_n}{2^n}\leq x\leq\frac{c_1}{2^1}+\frac{c_2}{2^2}\cdots+\frac{c_n+1}{2^n}.$$

- The two expansions are different only if x is one of the end points in these divisions, that is, if $x=\frac{m}{2^k}$ for some natural numbers m, k. Here without loss of generality we may take m to be odd.

Second Option in Binary expansion

- Now we explore the Second Option.
- We divide the interval $(0, 1]$ as $(0, \frac{1}{2}] \bigcup (\frac{1}{2}, 1]$.
- Then $(0, \frac{1}{2}]$ as $(0, \frac{1}{2^2}] \bigcup (\frac{1}{2^2}, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ as $(\frac{1}{2}, \frac{3}{2^2}] \bigcup (\frac{3}{2^2}, 1]$.
- This way we get a possibly new binary expansion, say the digits are $c_1, c_2, \ldots$, satisfying

$$\frac{c_1}{2^1} + \frac{c_2}{2^2} \cdots + \frac{c_n}{2^n} \leq x \leq \frac{c_1}{2^1} + \frac{c_2}{2^2} \cdots + \frac{c_n + 1}{2^n}.$$

- The two expansions are different only if x is one of the end points in these divisions, that is, if $x = \frac{m}{2^k}$ for some natural numbers m, k. Here without loss of generality we may take m to be odd.
- In other words in $(0, 1)$, only numbers of the form $\frac{m}{2^k}$, with natural numbers m, k have two binary expansions.
- For instance, $\frac{1}{2}$ is expressed as $0.10000000\ldots$ using the first option and as $0.0111111111\ldots$ through the second option.

Binary expansion continued

- Suppose $x \in (0, 1)$ is expressed using binary expansion, under either option, and $b_1, b_2, \ldots, b_n$ are the first n binary digits.

Binary expansion continued

- Suppose $x \in (0,1)$ is expressed using binary expansion, under either option, and $b_1, b_2, \ldots, b_n$ are the first n binary digits.
- Then

$$\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} \leq x \leq \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n + 1}{2^n}$$

Binary expansion continued

- Suppose $x \in (0,1)$ is expressed using binary expansion, under either option, and $b_1, b_2, \ldots, b_n$ are the first n binary digits.
- Then

$$\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} \leq x \leq \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n + 1}{2^n}$$

- From the proof of the nested intervals property, we see that

$$x = \sup\{\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} : n \in \mathbb{N}\}.$$

Binary expansion continued

- Suppose $x \in (0,1)$ is expressed using binary expansion, under either option, and $b_1, b_2, \ldots, b_n$ are the first n binary digits.
- Then

$$\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} \leq x \leq \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n + 1}{2^n}$$

- From the proof of the nested intervals property, we see that

$$x = \sup\{\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} : n \in \mathbb{N}\}.$$

- Note that

$$\begin{aligned} \tfrac{1}{2} &= \sup\{\tfrac{1}{2} + 0 + \cdots + 0(n-1 \text{ times }) : n \in \mathbb{N}\} \\ &= \sup\{0 + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^n} : n \in \mathbb{N}\}. \end{aligned}$$

Binary expansion continued

- Suppose $x \in (0,1)$ is expressed using binary expansion, under either option, and $b_1, b_2, \ldots, b_n$ are the first n binary digits.
- Then

$$\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} \leq x \leq \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n + 1}{2^n}$$

- From the proof of the nested intervals property, we see that

$$x = \sup\{\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} : n \in \mathbb{N}\}.$$

- Note that

$$\begin{aligned} \tfrac{1}{2} &= \sup\{\tfrac{1}{2} + 0 + \cdots + 0(n-1 \text{ times }) : n \in \mathbb{N}\} \\ &= \sup\{0 + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^n} : n \in \mathbb{N}\}. \end{aligned}$$

- Similarly $1 = \sup\{\frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^n} : n \in \mathbb{N}\}$.

Ternary and decimal expansions

- Similar to binary expansion we can have expansion with 'base' M, for any $M \in \{2, 3, 4, \ldots\}$, where we use only the digits $\{0, 1, 2, \ldots, (M-1)\}$.

Ternary and decimal expansions

- Similar to binary expansion we can have expansion with 'base' M, for any $M \in \{2, 3, 4, \ldots\}$, where we use only the digits $\{0, 1, 2, \ldots, (M-1)\}$.
- Ternary ($M = 3$) and decimal ($M = 10$)expansions are particularly useful.

Ternary and decimal expansions

- Similar to binary expansion we can have expansion with 'base' M, for any $M \in \{2, 3, 4, \ldots\}$, where we use only the digits $\{0, 1, 2, \ldots, (M-1)\}$.
- Ternary ($M = 3$) and decimal ($M = 10$)expansions are particularly useful.
- If $x = 0.d_1 d_2 \cdots$ is the decimal expansion of x, then, each $d_j \in \{0, 1, 2, \ldots, 9\}$ and

$$x = \sup\{\frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} : n \in \mathbb{N}\}.$$

Ternary and decimal expansions

- Similar to binary expansion we can have expansion with 'base' M, for any $M \in \{2, 3, 4, \ldots\}$, where we use only the digits $\{0, 1, 2, \ldots, (M-1)\}$.
- Ternary ($M = 3$) and decimal ($M = 10$)expansions are particularly useful.
- If $x = 0.d_1 d_2 \cdots$ is the decimal expansion of x, then, each $d_j \in \{0, 1, 2, \ldots, 9\}$ and

$$x = \sup\{\frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} : n \in \mathbb{N}\}.$$

- Here $x \in (0, 1)$ has two decimal expansions if and only if $x = \frac{m}{10^k}$ for some natural numbers m, k.

Ternary and decimal expansions

- Similar to binary expansion we can have expansion with 'base' M, for any $M \in \{2, 3, 4, \ldots\}$, where we use only the digits $\{0, 1, 2, \ldots, (M-1)\}$.
- Ternary ($M = 3$) and decimal ($M = 10$)expansions are particularly useful.
- If $x = 0.d_1 d_2 \cdots$ is the decimal expansion of x, then, each $d_j \in \{0, 1, 2, \ldots, 9\}$ and

$$x = \sup\{\frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} : n \in \mathbb{N}\}.$$

- Here $x \in (0, 1)$ has two decimal expansions if and only if $x = \frac{m}{10^k}$ for some natural numbers m, k.
- Alternatively x has two decimal expansions if and only if its decimal expansion is of the form $0.d_1 d_2 \ldots d_n 000000\ldots$ or it is of the form $0.d_1 d_2 \ldots d_n 999999\ldots$ for some d_j's.

Ternary and decimal expansions

- Similar to binary expansion we can have expansion with 'base' M, for any $M \in \{2, 3, 4, \ldots\}$, where we use only the digits $\{0, 1, 2, \ldots, (M-1)\}$.
- Ternary ($M = 3$) and decimal ($M = 10$)expansions are particularly useful.
- If $x = 0.d_1d_2\cdots$ is the decimal expansion of x, then, each $d_j \in \{0, 1, 2, \ldots, 9\}$ and

$$x = \sup\{\frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} : n \in \mathbb{N}\}.$$

- Here $x \in (0, 1)$ has two decimal expansions if and only if $x = \frac{m}{10^k}$ for some natural numbers m, k.
- Alternatively x has two decimal expansions if and only if its decimal expansion is of the form $0.d_1d_2\ldots d_n000000\ldots$ or it is of the form $0.d_1d_2\ldots d_n999999\ldots$ for some d_j's.
- In such cases, we say that x has a terminating decimal expansion. (It ends either with a sequence of 0's or with a sequence of 9's.)

Summary

- We summarize our observations as follows.

Summary

- We summarize our observations as follows.
- Theorem 12.2: Fix $M \in \mathbb{N}$ with $M \geq 2$. Then any real number $x \in [0,1)$ can be expressed as:

$$x = \sup\{\frac{d_1}{M} + \frac{d_2}{M^2} + \cdots + \frac{d_n}{M^n} : n \in \mathbb{N}, d_j \in \{0,1,\ldots,M-1\}\}$$

The sequence $d_1, d_2, \ldots$ is uniquely determined unless $x = \frac{m}{M^k}$ for some natural numbers m, k. Further, if $x = \frac{m}{M^k}$ then x has two possible expressions, one terminating with 0's and another terminating with $(M-1)$'s.

Summary

- We summarize our observations as follows.
- Theorem 12.2: Fix $M \in \mathbb{N}$ with $M \geq 2$. Then any real number $x \in [0,1)$ can be expressed as:

$$x = \sup\{\frac{d_1}{M} + \frac{d_2}{M^2} + \cdots + \frac{d_n}{M^n} : n \in \mathbb{N}, d_j \in \{0,1,\ldots,M-1\}\}$$

  The sequence $d_1, d_2, \ldots$ is uniquely determined unless $x = \frac{m}{M^k}$ for some natural numbers m, k. Further, if $x = \frac{m}{M^k}$ then x has two possible expressions, one terminating with 0's and another terminating with $(M-1)$'s.
- If $d_1, d_2, \ldots$ are as in this theorem, we say

$$x = 0.d_1 d_2 d_3 \ldots$$

  in base M.

Summary

- We summarize our observations as follows.
- Theorem 12.2: Fix $M \in \mathbb{N}$ with $M \geq 2$. Then any real number $x \in [0,1)$ can be expressed as:

$$x = \sup\{\frac{d_1}{M} + \frac{d_2}{M^2} + \cdots + \frac{d_n}{M^n} : n \in \mathbb{N}, d_j \in \{0, 1, \ldots, M-1\}\}$$

  The sequence $d_1, d_2, \ldots$ is uniquely determined unless $x = \frac{m}{M^k}$ for some natural numbers m, k. Further, if $x = \frac{m}{M^k}$ then x has two possible expressions, one terminating with 0's and another terminating with $(M-1)$'s.
- If $d_1, d_2, \ldots$ are as in this theorem, we say

$$x = 0.d_1 d_2 d_3 \ldots$$

  in base M.
- END OF LECTURE 12