ANALYSIS -I

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- Now consider $S \backslash \{x_1\}$. If $S \backslash \{x_1\}$ is empty, then $S = \{x_1\}$ and this would mean that S is finite. Therefore $S \backslash \{x_1\}$ is non-empty. Choose any $x_2 \in S \backslash \{x_1\}$.

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- Then by mathematical induction we have a sequence $\{x_1, x_2, \ldots\}$ of distinct elements in S. Clearly $T = \{x_n : n \in \mathbb{N}\}$ is equipotent with $\mathbb{N}$.

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- Conclude that $S \bigcup F$ is equipotent with S.

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- Clearly $T \bigcup C$ is equipotent with T.
- If $f : T \to T \bigcup C$ is a bijection, $\tilde{f} : S \to S \bigcup C$ defined by

$$(\tilde{f})(x) = \begin{cases} f(x) & x \in T; \\ x & x \in S \setminus T \end{cases}$$

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- Corollary 13.4: If S is an uncountable set and $T \subset S$ is countable then S is equipotent with $S \setminus T$.

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- Now $\mathbb{B} = A \bigcup B_0$. A is uncountable and B_0 is countable. Hence $\mathbb{B}$ is equipotent with A.
- Consequently $[0,1)$ and $\mathbb{B}$ are equipotent.

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- (iii) $[0,1]$ is equipotent with $[a,b]$ for any a,b in $\mathbb{R}$ with $a<b$: Consider the map $g:[0,1]\to[a,b]$ defined by

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- (v) It is an exercise to cover all the remaining cases.

More problems

- Show that $\mathbb{R} \times \mathbb{R}$ is equipotent with $\mathbb{R}$. More generally, show that $\mathbb{R}^n$ is equipotent with $\mathbb{R}$ for any $n \in \mathbb{N}$.

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