ANALYSIS -I

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

14. Direct and inverse images of functions

- Notation: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. The for $A \subseteq X$, $f(A)$ is defined as:

14. Direct and inverse images of functions

- Notation: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. The for $A \subseteq X$, $f(A)$ is defined as:
- $$f(A) := \{f(x) : x \in A\}.$$
- Example 14.1: Suppose $X = \{1, 2, 3\}$ and $Y = \{u, v, w\}$ and $f : X \to Y$ is defined by $f(1) = f(2) = u$ and $f(3) = v$.

14. Direct and inverse images of functions

- Notation: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. The for $A \subseteq X$, $f(A)$ is defined as:
- $$f(A) := \{f(x) : x \in A\}.$$
- Example 14.1: Suppose $X = \{1, 2, 3\}$ and $Y = \{u, v, w\}$ and $f : X \to Y$ is defined by $f(1) = f(2) = u$ and $f(3) = v$.
- Then $f(\{1, 2\}) = \{u\}$ and $f(\{3\}) = \{v\}$.

14. Direct and inverse images of functions

- Notation: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. The for $A \subseteq X$, $f(A)$ is defined as:
- $$f(A) := \{f(x) : x \in A\}.$$
- Example 14.1: Suppose $X = \{1, 2, 3\}$ and $Y = \{u, v, w\}$ and $f : X \to Y$ is defined by $f(1) = f(2) = u$ and $f(3) = v$.
- Then $f(\{1, 2\}) = \{u\}$ and $f(\{3\}) = \{v\}$.
- Here we have slight abuse of notation as we are defining $f(A)$ for subsets of X and not elements of X, where as, normally when we write $f(x)$, x is an element of X. However, this notation is standard.

14. Direct and inverse images of functions

- Notation: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. The for $A \subseteq X$, $f(A)$ is defined as:
- $$f(A) := \{f(x) : x \in A\}.$$
- Example 14.1: Suppose $X = \{1, 2, 3\}$ and $Y = \{u, v, w\}$ and $f : X \to Y$ is defined by $f(1) = f(2) = u$ and $f(3) = v$.
- Then $f(\{1, 2\}) = \{u\}$ and $f(\{3\}) = \{v\}$.
- Here we have slight abuse of notation as we are defining $f(A)$ for subsets of X and not elements of X, where as, normally when we write $f(x)$, x is an element of X. However, this notation is standard.
- Note that for any element x of X, $f(\{x\}) = \{f(x)\}$, which is the singleton set containing $f(x)$ and is different from the element $f(x)$. This distinction between elements and singleton sets should always be maintained to avoid confusion.

Basic properties

- Proposition 14.2: Let $f : X \to Y$ be a function. Then,

Basic properties

- Proposition 14.2: Let $f : X \to Y$ be a function. Then,
- (i) $f(\emptyset) = \emptyset$.

Basic properties

- Proposition 14.2: Let $f : X \to Y$ be a function. Then,
- (i) $f(\emptyset) = \emptyset$.
- (ii) In general, $f(X) \neq Y$.

Basic properties

- Proposition 14.2: Let $f : X \to Y$ be a function. Then,
- (i) $f(\emptyset) = \emptyset$.
- (ii) In general, $f(X) \neq Y$.
- (iii) In general, for $A, B \subseteq X$,

$$f(A \bigcap B) \neq f(A) \bigcap f(B).$$

Basic properties

- Proposition 14.2: Let $f : X \to Y$ be a function. Then,
- (i) $f(\emptyset) = \emptyset$.
- (ii) In general, $f(X) \neq Y$.
- (iii) In general, for $A, B \subseteq X$,

$$f(A \bigcap B) \neq f(A) \bigcap f(B).$$

- (iv) For any two subsets A, B of X,

$$f(A \bigcup B) = f(A) \bigcup f(B).$$

- More generally, for arbitrary family $\{A_i : i \in I\}$ of subsets of X,

$$f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i).$$

Basic properties

- Proposition 14.2: Let $f : X \to Y$ be a function. Then,
- (i) $f(\emptyset) = \emptyset$.
- (ii) In general, $f(X) \neq Y$.
- (iii) In general, for $A, B \subseteq X$,

$$f(A \bigcap B) \neq f(A) \bigcap f(B).$$

- (iv) For any two subsets A, B of X,

$$f(A \bigcup B) = f(A) \bigcup f(B).$$

- More generally, for arbitrary family $\{A_i : i \in I\}$ of subsets of X,

$$f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i).$$

- (v) In general, for $A \subseteq X$

$$f(A^c) \neq (f(A))^c.$$

Examples

- Example 14.3: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2, \quad \forall x \in \mathbb{R}$.

Examples

- Example 14.3: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2, \quad \forall x \in \mathbb{R}$.
- Take $A = (-\infty, 0]$ and $B = [0, \infty)$. Then

Examples

- Example 14.3: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2, \quad \forall x \in \mathbb{R}$.
- Take $A = (-\infty, 0]$ and $B = [0, \infty)$. Then
- $A \bigcap B = \{0\}$.

Examples

- Example 14.3: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2, \quad \forall x \in \mathbb{R}$.
- Take $A = (-\infty, 0]$ and $B = [0, \infty)$. Then
- $A \bigcap B = \{0\}$.
- $f(A) \bigcap f(B) = [0, \infty) \bigcap [0, \infty) = [0, \infty)$, where as,

Examples

- Example 14.3: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2, \quad \forall x \in \mathbb{R}$.
- Take $A = (-\infty, 0]$ and $B = [0, \infty)$. Then
- $A \bigcap B = \{0\}$.
- $f(A) \bigcap f(B) = [0, \infty) \bigcap [0, \infty) = [0, \infty)$, where as,
- $f(A \bigcap B) = f(\{0\}) = \{0\}$.

Examples

- Example 14.3: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2, \quad \forall x \in \mathbb{R}$.
- Take $A = (-\infty, 0]$ and $B = [0, \infty)$. Then
- $A \bigcap B = \{0\}$.
- $f(A) \bigcap f(B) = [0, \infty) \bigcap [0, \infty) = [0, \infty)$, where as,
- $f(A \bigcap B) = f(\{0\}) = \{0\}$.
- Hence $f(A \bigcap B) \neq f(A) \bigcap f(B)$.

Continuation

- The prof of Proposition 14.2 is an exercise.

Continuation

- The prof of Proposition 14.2 is an exercise.
- For instance, if $y \in f(A \bigcup B)$, then $y = f(x)$ for some $x \in A \bigcup B$. Here either $x \in A$ or $x \in B$ (or both). If $x \in A$, we get $y \in f(A)$. If $x \in B$, we get $y \in f(B)$. Consequently, we get $y \in f(A) \bigcup f(B)$. This shows that $f(A \bigcup B) \subseteq f(A) \bigcup f(B)$.

Continuation

- The prof of Proposition 14.2 is an exercise.
- For instance, if $y \in f(A \bigcup B)$, then $y = f(x)$ for some $x \in A \bigcup B$. Here either $x \in A$ or $x \in B$ (or both). If $x \in A$, we get $y \in f(A)$. If $x \in B$, we get $y \in f(B)$. Consequently, we get $y \in f(A) \bigcup f(B)$. This shows that $f(A \bigcup B) \subseteq f(A) \bigcup f(B)$.
- Similarly, you can show $f(A) \bigcup f(B) \subseteq f(A \bigcup B)$ and conclude that $f(A \bigcup B) = f(A) \bigcup f(B)$.

Characterizations

- Theorem 14.4: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.

Characterizations

- Theorem 14.4: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.
- (a) $f(X) = Y$ if and only if f is surjective.

Characterizations

- Theorem 14.4: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.
- (a) $f(X) = Y$ if and only if f is surjective.
- (b) $f(A \bigcap B) = f(A) \bigcap f(B)$ for all subsets A, B of X if and only if f is injective.

Characterizations

- Theorem 14.4: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.
- (a) $f(X) = Y$ if and only if f is surjective.
- (b) $f(A \bigcap B) = f(A) \bigcap f(B)$ for all subsets A, B of X if and only if f is injective.
- (c) $f(A^c) = (f(A))^c$ for all subsets A of X if and only if f is a bijection.

Characterizations

- Theorem 14.4: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.
- (a) $f(X) = Y$ if and only if f is surjective.
- (b) $f(A \bigcap B) = f(A) \bigcap f(B)$ for all subsets A, B of X if and only if f is injective.
- (c) $f(A^c) = (f(A))^c$ for all subsets A of X if and only if f is a bijection.
- Proof: (a) follows from the definition of surjectivity. (b) and (c) are interesting exercises.

Inverse images

- Notation: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then for any subset V of Y,

$$f^{-1}(V) := \{x \in X : f(x) \in V\}.$$

Inverse images

- Notation: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then for any subset V of Y,

$$f^{-1}(V) := \{x \in X : f(x) \in V\}.$$

- For instance, for $f : \{1, 2, 3\} \to \{u, v, w\}$ defined by $f(1) = f(2) = u$ and $f(3) = v$,

$$f^{-1}(\{u\}) = \{1, 2\}, \quad f^{-1}(\{w\}) = \emptyset.$$

Inverse images

- Notation: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then for any subset V of Y,

$$f^{-1}(V) := \{x \in X : f(x) \in V\}.$$

- For instance, for $f : \{1, 2, 3\} \to \{u, v, w\}$ defined by $f(1) = f(2) = u$ and $f(3) = v$,

$$f^{-1}(\{u\}) = \{1, 2\}, \quad f^{-1}(\{w\}) = \emptyset.$$

- Here also there is some abuse of notation as we writing f^{-1} even when f is not invertible. But we are defining f^{-1} for subsets of Y and not for elements of Y.

Inverse images

- Notation: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then for any subset V of Y,

$$f^{-1}(V) := \{x \in X : f(x) \in V\}.$$

- For instance, for $f : \{1, 2, 3\} \to \{u, v, w\}$ defined by $f(1) = f(2) = u$ and $f(3) = v$,

$$f^{-1}(\{u\}) = \{1, 2\}, \quad f^{-1}(\{w\}) = \emptyset.$$

- Here also there is some abuse of notation as we writing f^{-1} even when f is not invertible. But we are defining f^{-1} for subsets of Y and not for elements of Y.
- For the example, $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = x^2, \;\; \forall x \in \mathbb{R}$, we see that $g^{-1}(\{0\}) = \{0\}$ and $g^{-1}([0, \infty)) = \mathbb{R}$.

Basic properties of inverse images

- Theorem 14.5: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then following properties hold.

Basic properties of inverse images

- Theorem 14.5: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then following properties hold.
- (i) $f^{-1}(\emptyset) = \emptyset$;

Basic properties of inverse images

- Theorem 14.5: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then following properties hold.
- (i) $f^{-1}(\emptyset) = \emptyset$;
- (ii) $f^{-1}(Y) = X$;

Basic properties of inverse images

- Theorem 14.5: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then following properties hold.
- (i) $f^{-1}(\emptyset) = \emptyset$;
- (ii) $f^{-1}(Y) = X$;
- (iii) $f^{-1}(V \bigcap W) = f^{-1}(V) \bigcap f^{-1}(W)$ for subsets V, W of Y. More generally, for any arbitrary collection $\{V_i : i \in I\}$ of subsets of Y,
$$f^{-1}(\bigcap_{i \in I} V_i) = \bigcap_{i \in I} f^{-1}(V_i).$$

Basic properties of inverse images

- Theorem 14.5: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then following properties hold.
- (i) $f^{-1}(\emptyset) = \emptyset$;
- (ii) $f^{-1}(Y) = X$;
- (iii) $f^{-1}(V \bigcap W) = f^{-1}(V) \bigcap f^{-1}(W)$ for subsets V, W of Y. More generally, for any arbitrary collection $\{V_i : i \in I\}$ of subsets of Y,
$$f^{-1}(\bigcap_{i \in I} V_i) = \bigcap_{i \in I} f^{-1}(V_i).$$
- (iv) $f^{-1}(V \bigcup W) = f^{-1}(V) \bigcup f^{-1}(W)$ for subsets V, W of Y. More generally, for any arbitrary collection $\{V_i : i \in I\}$ of subsets of Y,
$$f^{-1}(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} f^{-1}(V_i).$$

Basic properties of inverse images

- Theorem 14.5: Let X, Y be non-empty sets and let $f : X \to Y$ be a function. Then following properties hold.
- (i) $f^{-1}(\emptyset) = \emptyset$;
- (ii) $f^{-1}(Y) = X$;
- (iii) $f^{-1}(V \bigcap W) = f^{-1}(V) \bigcap f^{-1}(W)$ for subsets V, W of Y. More generally, for any arbitrary collection $\{V_i : i \in I\}$ of subsets of Y,
$$f^{-1}(\bigcap_{i \in I} V_i) = \bigcap_{i \in I} f^{-1}(V_i).$$
- (iv) $f^{-1}(V \bigcup W) = f^{-1}(V) \bigcup f^{-1}(W)$ for subsets V, W of Y. More generally, for any arbitrary collection $\{V_i : i \in I\}$ of subsets of Y,
$$f^{-1}(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} f^{-1}(V_i).$$
- (v) $f^{-1}(V^c) = (f^{-1}(V))^c$ for every subset V of Y.

Continuation

- It is indeed amazing that the inverse image f^{-1} respects all set theoretic operations with no conditions imposed on f. This is a very useful fact to remember.

Continuation

- It is indeed amazing that the inverse image f^{-1} respects all set theoretic operations with no conditions imposed on f. This is a very useful fact to remember.
- The proof of Theorem 14.5 is also as an exercise.

Continuation

- It is indeed amazing that the inverse image f^{-1} respects all set theoretic operations with no conditions imposed on f. This is a very useful fact to remember.
- The proof of Theorem 14.5 is also as an exercise.
- Theorem 14.6: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.

Continuation

- It is indeed amazing that the inverse image f^{-1} respects all set theoretic operations with no conditions imposed on f. This is a very useful fact to remember.
- The proof of Theorem 14.5 is also as an exercise.
- Theorem 14.6: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.
- (a) For any subset A of X,

$$f^{-1}(f(A)) \supseteq A$$

and the equality may not hold.

Continuation

- It is indeed amazing that the inverse image f^{-1} respects all set theoretic operations with no conditions imposed on f. This is a very useful fact to remember.
- The proof of Theorem 14.5 is also as an exercise.
- Theorem 14.6: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.
- (a) For any subset A of X,

$$f^{-1}(f(A)) \supseteq A$$

  and the equality may not hold.
- (b) For any subset V of Y,

$$f(f^{-1}(V)) \subseteq V$$

  and the equality may not hold.

Continuation

- It is indeed amazing that the inverse image f^{-1} respects all set theoretic operations with no conditions imposed on f. This is a very useful fact to remember.
- The proof of Theorem 14.5 is also as an exercise.
- Theorem 14.6: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.
- (a) For any subset A of X,

$$f^{-1}(f(A)) \supseteq A$$

and the equality may not hold.
- (b) For any subset V of Y,

$$f(f^{-1}(V)) \subseteq V$$

and the equality may not hold.
- Proof: Exercise.

Continuation

- It is indeed amazing that the inverse image f^{-1} respects all set theoretic operations with no conditions imposed on f. This is a very useful fact to remember.
- The proof of Theorem 14.5 is also as an exercise.
- Theorem 14.6: Let X, Y be non-empty sets and let $f : X \to Y$ be a function.
- (a) For any subset A of X,

$$f^{-1}(f(A)) \supseteq A$$

  and the equality may not hold.
- (b) For any subset V of Y,

$$f(f^{-1}(V)) \subseteq V$$

  and the equality may not hold.
- Proof: Exercise.
- END OF LECTURE 14.