ANALYSIS -I

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$$|x - y| \leq |x - z| + |z - y|.$$
- We have already proved these results.

Definition and Examples

- Definition 15.1 : A sequence of real numbers

$$a_1, a_2, a_3, \ldots$$

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- Example 15.2: Consider the function $a : \mathbb{N} \to \mathbb{N}$ defined by $a(n) = n^2$, this gives us the sequence,

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- Example 15.3 (Fibonacci sequence): This is the sequence:

$$1, 1, 2, 3, 5, 8, \ldots,$$

  defined 'recursively', by $a_1 = 1, a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$.

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$$|a_n - x| < \epsilon, \quad \forall n \geq K.$$

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- A sequence which is not convergent is said to be divergent.
- We may write, $|a_n - x| < \epsilon$, equivalently as $x - \epsilon < a_n < x + \epsilon$ or as $a_n \in (x - \epsilon, x + \epsilon)$.

Constant sequence

- Example 15.3 (Constant sequence): Choose and fix a real number c. Let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence defined by $a_n = c, \quad \forall n \in \mathbb{N}$. So it is the sequence:

$$c, c, c, c, \ldots$$

Then $\{a_n\}_{n\in\mathbb{N}}$ is convergent and it converges to c.

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- Hence $\{a_n\}_{n\in\mathbb{N}}$ converges to c.

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- Choose any $n \geq \max\{K_1, K_2\}$. Then both the previous inequalities are true. Then by triangle inequality we get

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- Hence

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- Since $\inf\{\epsilon : \epsilon > 0\} = 0$, we get $0 \leq \frac{1}{2}|x - y| \leq 0$,
- Hence $\frac{1}{2}|x - y| = 0$ or $|x - y| = 0$, which is same as saying $x = y$.

Notation

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- So the convergence or non-convergence is a property of the whole sequence.

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- Then for any $n \geq K$, we have $\frac{1}{n} \leq \frac{1}{K} < \epsilon$. Hence,
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- We may also write this as: $\lim_{n\to\infty} \frac{1}{n} = 0$.

Boundedness

- Definition 15.7: A sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that

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- Similarly choosing an even number $n \geq K$, we get $|1 - x| < \epsilon$.

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- END OF LECTURE 15