ANALYSIS -I

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 16. Some limit theorems

- Recall the definition of convergence:

Lecture 16. Some limit theorems

- Recall the definition of convergence:
- Definition 15.2: A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is said to be convergent if there exists a real number x, where for every $\epsilon > 0$, there exists a natural number K (depending upon ϵ) such that

$$|a_n - x| < \epsilon, \quad \forall n \geq K.$$

  In such a case, $\{a_n\}_{n\in\mathbb{N}}$ is said to converge to x, and x is said to be the limit of $\{a_n\}_{n\in\mathbb{N}}$.
- Notation: If $\{a_n\}_{n\in\mathbb{N}}$ converges to x, we write

$$\lim_{n\to\infty} a_n = x.$$

Lecture 16. Some limit theorems

- Recall the definition of convergence:
- Definition 15.2: A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is said to be convergent if there exists a real number x, where for every $\epsilon > 0$, there exists a natural number K (depending upon ϵ) such that

$$|a_n - x| < \epsilon, \quad \forall n \geq K.$$

  In such a case, $\{a_n\}_{n\in\mathbb{N}}$ is said to converge to x, and x is said to be the limit of $\{a_n\}_{n\in\mathbb{N}}$.
- Notation: If $\{a_n\}_{n\in\mathbb{N}}$ converges to x, we write

$$\lim_{n\to\infty} a_n = x.$$

- A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to be bounded if there exists positive real number M such that

$$|a_n| \leq M, \quad \forall n \in \mathbb{N}.$$

Lecture 16. Some limit theorems

- Recall the definition of convergence:
- Definition 15.2: A sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is said to be convergent if there exists a real number x, where for every $\epsilon > 0$, there exists a natural number K (depending upon ϵ) such that
$$|a_n - x| < \epsilon, \quad \forall n \geq K.$$
In such a case, $\{a_n\}_{n\in\mathbb{N}}$ is said to converge to x, and x is said to be the limit of $\{a_n\}_{n\in\mathbb{N}}$.
- Notation: If $\{a_n\}_{n\in\mathbb{N}}$ converges to x, we write
$$\lim_{n\to\infty} a_n = x.$$
- A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to be bounded if there exists positive real number M such that
$$|a_n| \leq M, \quad \forall n \in \mathbb{N}.$$
- We have seen that every convergent sequence is bounded but the converse is not true.

Product with a bounded sequence

- Theorem 16.1: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence converging to 0 and $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence then $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.

Product with a bounded sequence

- Theorem 16.1: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence converging to 0 and $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence then $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.
- Proof: As $\{b_n\}_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that

$$|b_n| \leq M, \quad \forall n \in \mathbb{N}.$$

Product with a bounded sequence

- Theorem 16.1: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence converging to 0 and $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence then $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.
- Proof: As $\{b_n\}_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that

$$|b_n| \leq M, \quad \forall n \in \mathbb{N}.$$

- For $\epsilon > 0$, take $\epsilon' = \frac{\epsilon}{M}$.

Product with a bounded sequence

- Theorem 16.1: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence converging to 0 and $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence then $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.
- Proof: As $\{b_n\}_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that

$$|b_n| \leq M, \quad \forall n \in \mathbb{N}.$$

- For $\epsilon > 0$, take $\epsilon' = \frac{\epsilon}{M}$.
- As $\epsilon' > 0$, and $\{a_n\}_{n\in\mathbb{N}}$ converges to 0, there exists a natural number K such that

$$|a_n - 0| < \epsilon', \;\; \forall n \geq K.$$

Product with a bounded sequence

- Theorem 16.1: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence converging to 0 and $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence then $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.
- Proof: As $\{b_n\}_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that

$$|b_n| \leq M, \quad \forall n \in \mathbb{N}.$$

- For $\epsilon > 0$, take $\epsilon' = \frac{\epsilon}{M}$.
- As $\epsilon' > 0$, and $\{a_n\}_{n\in\mathbb{N}}$ converges to 0, there exists a natural number K such that

$$|a_n - 0| < \epsilon', \quad \forall n \geq K.$$

- Now for $n \geq K$,

$$|a_nb_n - 0| = |a_nb_n| = |a_n||b_n| \leq |a_n|M < \epsilon'.M = \epsilon.$$

Product with a bounded sequence

- Theorem 16.1: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence converging to 0 and $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence then $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.
- Proof: As $\{b_n\}_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that

$$|b_n| \leq M, \quad \forall n \in \mathbb{N}.$$

- For $\epsilon > 0$, take $\epsilon' = \frac{\epsilon}{M}$.
- As $\epsilon' > 0$, and $\{a_n\}_{n\in\mathbb{N}}$ converges to 0, there exists a natural number K such that

$$|a_n - 0| < \epsilon', \ \ \forall n \geq K.$$

- Now for $n \geq K$,

$$|a_nb_n - 0| = |a_nb_n| = |a_n||b_n| \leq |a_n|M < \epsilon'.M = \epsilon.$$

- Hence $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.

Product with a bounded sequence

- Theorem 16.1: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence converging to 0 and $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence then $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.
- Proof: As $\{b_n\}_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that

$$|b_n| \leq M, \quad \forall n \in \mathbb{N}.$$

- For $\epsilon > 0$, take $\epsilon' = \frac{\epsilon}{M}$.
- As $\epsilon' > 0$, and $\{a_n\}_{n\in\mathbb{N}}$ converges to 0, there exists a natural number K such that

$$|a_n - 0| < \epsilon', \;\; \forall n \geq K.$$

- Now for $n \geq K$,

$$|a_nb_n - 0| = |a_nb_n| = |a_n||b_n| \leq |a_n|M < \epsilon'.M = \epsilon.$$

- Hence $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.
- Taking $a_n = \frac{1}{n}$ and $b_n = n$, we see that the result may not be true when $\{b_n\}_{n\in\mathbb{N}}$ is not bounded.

Sums and products of sequences

- Theorem 16.2: Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences converging to x, y respectively.

Sums and products of sequences

- Theorem 16.2: Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences converging to x, y respectively.
- (a) For $c \in \mathbb{R}$, $\{ca_n\}_{n\in\mathbb{N}}$ converges to cx.

Sums and products of sequences

- Theorem 16.2: Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences converging to x, y respectively.
- (a) For $c \in \mathbb{R}$, $\{ca_n\}_{n\in\mathbb{N}}$ converges to cx.
- (b) $\{a_n + b_n\}_{n\in\mathbb{N}}$ converges to $x + y$.

Sums and products of sequences

- Theorem 16.2: Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences converging to x, y respectively.
- (a) For $c \in \mathbb{R}$, $\{ca_n\}_{n\in\mathbb{N}}$ converges to cx.
- (b) $\{a_n + b_n\}_{n\in\mathbb{N}}$ converges to $x + y$.
- (c) For $c, d \in \mathbb{R}$, $\{ca_n + db_n\}_{n\in\mathbb{N}}$ converges to $cx + dy$.
- (d) $\{a_n b_n\}_{n\in\mathbb{N}}$ converges to xy.

Sums and products of sequences

- Theorem 16.2: Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences converging to x, y respectively.
- (a) For $c \in \mathbb{R}$, $\{ca_n\}_{n\in\mathbb{N}}$ converges to cx.
- (b) $\{a_n + b_n\}_{n\in\mathbb{N}}$ converges to $x + y$.
- (c) For $c, d \in \mathbb{R}$, $\{ca_n + db_n\}_{n\in\mathbb{N}}$ converges to $cx + dy$.
- (d) $\{a_n b_n\}_{n\in\mathbb{N}}$ converges to xy.
- (e) If $b_n \neq 0$ for every $n \in \mathbb{N}$ and $y \neq 0$ then $\{\frac{a_n}{b_n}\}_{n\in\mathbb{N}}$ converges to $\frac{x}{y}$.

Proof of (a)

- Proof: (a) Clearly the result is true if $c = 0$. So assume that $c \neq 0$.

Proof of (a)

- Proof: (a) Clearly the result is true if $c = 0$. So assume that $c \neq 0$.
- Now for $\epsilon > 0$, take $\epsilon' = \frac{\epsilon}{|c|} > 0$.

Proof of (a)

- Proof: (a) Clearly the result is true if $c = 0$. So assume that $c \neq 0$.
- Now for $\epsilon > 0$, take $\epsilon' = \frac{\epsilon}{|c|} > 0$.
- As $\{a_n\}_{n \in \mathbb{N}}$ converges to x, there exists $K \in \mathbb{N}$ such that

$$|a_n - x| < \epsilon', \quad \forall n \geq K.$$

Proof of (a)

- Proof: (a) Clearly the result is true if $c = 0$. So assume that $c \neq 0$.
- Now for $\epsilon > 0$, take $\epsilon' = \frac{\epsilon}{|c|} > 0$.
- As $\{a_n\}_{n\in\mathbb{N}}$ converges to x, there exists $K \in \mathbb{N}$ such that

$$|a_n - x| < \epsilon', \quad \forall n \geq K.$$

- Then for $n \geq K$,

$$|ca_n - cx| = |c||a_n - x| < |c|\epsilon' = \epsilon.$$

Proof of (a)

- Proof: (a) Clearly the result is true if $c = 0$. So assume that $c \neq 0$.
- Now for $\epsilon > 0$, take $\epsilon' = \frac{\epsilon}{|c|} > 0$.
- As $\{a_n\}_{n \in \mathbb{N}}$ converges to x, there exists $K \in \mathbb{N}$ such that

$$|a_n - x| < \epsilon', \quad \forall n \geq K.$$

- Then for $n \geq K$,

$$|ca_n - cx| = |c||a_n - x| < |c|\epsilon' = \epsilon.$$

- Hence $\{ca_n\}_{n \in \mathbb{N}}$ converges to cx.

Proof of (b) and (c)

- For $\epsilon > 0$, we have $\frac{\epsilon}{2} > 0$. Choose K_1 such that

$$|a_n - x| < \frac{\epsilon}{2}, \quad \forall n \geq K_1.$$

Proof of (b) and (c)

- For $\epsilon > 0$, we have $\frac{\epsilon}{2} > 0$. Choose K_1 such that

$$|a_n - x| < \frac{\epsilon}{2}, \quad \forall n \geq K_1.$$

- Choose K_2 such that

$$|b_n - y| < \frac{\epsilon}{2}, \quad \forall n \geq K_2.$$

Proof of (b) and (c)

- For $\epsilon > 0$, we have $\frac{\epsilon}{2} > 0$. Choose K_1 such that

$$|a_n - x| < \frac{\epsilon}{2}, \quad \forall n \geq K_1.$$

- Choose K_2 such that

$$|b_n - y| < \frac{\epsilon}{2}, \quad \forall n \geq K_2.$$

- Take $K = \max\{K_1, K_2\}$.

Proof of (b) and (c)

- For $\epsilon > 0$, we have $\frac{\epsilon}{2} > 0$. Choose K_1 such that

$$|a_n - x| < \frac{\epsilon}{2}, \quad \forall n \geq K_1.$$

- Choose K_2 such that

$$|b_n - y| < \frac{\epsilon}{2}, \quad \forall n \geq K_2.$$

- Take $K = \max\{K_1, K_2\}$.
- Then for $n \geq K$,

$$|(a_n + b_n) - (x + y)| \leq |a_n - x| + |b_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Proof of (b) and (c)

- For $\epsilon > 0$, we have $\frac{\epsilon}{2} > 0$. Choose K_1 such that

$$|a_n - x| < \frac{\epsilon}{2}, \quad \forall n \geq K_1.$$

- Choose K_2 such that

$$|b_n - y| < \frac{\epsilon}{2}, \quad \forall n \geq K_2.$$

- Take $K = \max\{K_1, K_2\}$.
- Then for $n \geq K$,

$$|(a_n + b_n) - (x + y)| \leq |a_n - x| + |b_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

- Hence $\{a_n + b_n\}_{n \in \mathbb{N}}$ converges to $x + y$.

Proof of (b) and (c)

- For $\epsilon > 0$, we have $\frac{\epsilon}{2} > 0$. Choose K_1 such that
$$|a_n - x| < \frac{\epsilon}{2}, \quad \forall n \geq K_1.$$
- Choose K_2 such that
$$|b_n - y| < \frac{\epsilon}{2}, \quad \forall n \geq K_2.$$
- Take $K = \max\{K_1, K_2\}$.
- Then for $n \geq K$,
$$|(a_n + b_n) - (x + y)| \leq |a_n - x| + |b_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
- Hence $\{a_n + b_n\}_{n \in \mathbb{N}}$ converges to $x + y$.
- Clearly (c) follows from (a) and (b).

Proof of (d)

- Now we need to estimate $|a_n b_n - xy|$.

Proof of (d)

- Now we need to estimate $|a_n b_n - xy|$.
- By triangle inequality,

$$\begin{aligned} |a_n b_n - xy| &\leq |a_n b_n - x b_n + x b_n - xy| \\ &\leq |(a_n - x) b_n| + |x(b_n - y)| \\ &\leq |a_n - x||b_n| + |x||b_n - y|. \end{aligned}$$

Proof of (d)

- Now we need to estimate $|a_n b_n - xy|$.
- By triangle inequality,

$$\begin{aligned} |a_n b_n - xy| &\leq |a_n b_n - x b_n + x b_n - xy| \\ &\leq |(a_n - x) b_n| + |x(b_n - y)| \\ &\leq |a_n - x||b_n| + |x||b_n - y|. \end{aligned}$$

- As $\{b_n\}_{n\in\mathbb{N}}$ is convergent it is a bounded sequence. Hence there exists $M > 0$ such that $|b_n| \leq M$ for all n.

Proof of (d)

- Now we need to estimate $|a_n b_n - xy|$.
- By triangle inequality,

$$\begin{aligned} |a_n b_n - xy| &\leq |a_n b_n - x b_n + x b_n - xy| \\ &\leq |(a_n - x) b_n| + |x(b_n - y)| \\ &\leq |a_n - x||b_n| + |x||b_n - y|. \end{aligned}$$

- As $\{b_n\}_{n \in \mathbb{N}}$ is convergent it is a bounded sequence. Hence there exists $M > 0$ such that $|b_n| \leq M$ for all n.
- For $\epsilon > 0$, choose $K_1 \in \mathbb{N}$ such that

$$|a_n - x| < \frac{\epsilon}{2M}, \quad \forall n \geq K_1.$$

Proof of (d)

- Now we need to estimate $|a_n b_n - xy|$.
- By triangle inequality,

$$\begin{aligned} |a_n b_n - xy| &\leq |a_n b_n - xb_n + xb_n - xy| \\ &\leq |(a_n - x)b_n| + |x(b_n - y)| \\ &\leq |a_n - x||b_n| + |x||b_n - y|. \end{aligned}$$

- As $\{b_n\}_{n\in\mathbb{N}}$ is convergent it is a bounded sequence. Hence there exists $M > 0$ such that $|b_n| \leq M$ for all n.
- For $\epsilon > 0$, choose $K_1 \in \mathbb{N}$ such that

$$|a_n - x| < \frac{\epsilon}{2M}, \quad \forall n \geq K_1.$$

- Choose $K_2 \in \mathbb{N}$ such that

$$|x||b_n - y| < \frac{\epsilon}{2}, \quad \forall n \geq K_2.$$

Proof of (d)

- Now we need to estimate $|a_n b_n - xy|$.
- By triangle inequality,

$$\begin{aligned} |a_n b_n - xy| &\leq |a_n b_n - x b_n + x b_n - xy| \\ &\leq |(a_n - x) b_n| + |x(b_n - y)| \\ &\leq |a_n - x||b_n| + |x||b_n - y|. \end{aligned}$$

- As $\{b_n\}_{n\in\mathbb{N}}$ is convergent it is a bounded sequence. Hence there exists $M > 0$ such that $|b_n| \leq M$ for all n.
- For $\epsilon > 0$, choose $K_1 \in \mathbb{N}$ such that

$$|a_n - x| < \frac{\epsilon}{2M}, \quad \forall n \geq K_1.$$

- Choose $K_2 \in \mathbb{N}$ such that

$$|x||b_n - y| < \frac{\epsilon}{2}, \quad \forall n \geq K_2.$$

- If $x \neq 0$, this can be done by taking $\epsilon' = \frac{\epsilon}{2|x|}$, and using convergence of $\{b_n\}$. If $x = 0$, the inequality is trivially true and we can simply take $K_2 = 1$.

Continuation

- Now for $n \geq \max\{K_1, K_2\}$

$$\begin{aligned} |a_n b_n - xy| &\leq |a_n - x||b_n| + |x||b_n - y| \\ &< \frac{\epsilon}{2M}.M + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Continuation

- Now for $n \geq \max\{K_1, K_2\}$

$$\begin{aligned} |a_n b_n - xy| &\leq |a_n - x||b_n| + |x||b_n - y| \\ &< \frac{\epsilon}{2M}.M + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

- Hence $\{a_n b_n\}_{n\in\mathbb{N}}$ converges to xy.

Proof of (e)

- Clearly (e) follows from (d) if we show that $\frac{1}{b_n}$ converges to $\frac{1}{y}$. (Note that here we are assuming that $b_n \neq 0$ for every n and $y \neq 0$.)

Proof of (e)

- Clearly (e) follows from (d) if we show that $\frac{1}{b_n}$ converges to $\frac{1}{y}$. (Note that here we are assuming that $b_n \neq 0$ for every n and $y \neq 0$.)
- Now we need to estimate,

$$|\frac{1}{b_n} - \frac{1}{y}| = |\frac{y - b_n}{b_n y}| = \frac{|y - b_n|}{|b_n y|}.$$

Proof of (e)

- Clearly (e) follows from (d) if we show that $\frac{1}{b_n}$ converges to $\frac{1}{y}$. (Note that here we are assuming that $b_n \neq 0$ for every n and $y \neq 0$.)
- Now we need to estimate,

$$|\frac{1}{b_n} - \frac{1}{y}| = |\frac{y - b_n}{b_n y}| = \frac{|y - b_n|}{|b_n y|}.$$

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.

Proof of (e)

- Clearly (e) follows from (d) if we show that $\frac{1}{b_n}$ converges to $\frac{1}{y}$. (Note that here we are assuming that $b_n \neq 0$ for every n and $y \neq 0$.)
- Now we need to estimate,

$$|\frac{1}{b_n} - \frac{1}{y}| = |\frac{y - b_n}{b_n y}| = \frac{|y - b_n|}{|b_n y|}.$$

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Once we prove this claim, for $\epsilon > 0$, take $\epsilon' = \frac{\epsilon |y|}{M}$, and choose $K \in \mathbb{N}$ such that

$$|b_n - y| < \epsilon', \quad \forall n \geq K.$$

Proof of (e)

- Clearly (e) follows from (d) if we show that $\frac{1}{b_n}$ converges to $\frac{1}{y}$. (Note that here we are assuming that $b_n \neq 0$ for every n and $y \neq 0$.)
- Now we need to estimate,

$$|\frac{1}{b_n} - \frac{1}{y}| = |\frac{y - b_n}{b_n y}| = \frac{|y - b_n|}{|b_n y|}.$$

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Once we prove this claim, for $\epsilon > 0$, take $\epsilon' = \frac{\epsilon|y|}{M}$, and choose $K \in \mathbb{N}$ such that

$$|b_n - y| < \epsilon', \quad \forall n \geq K.$$

- Then for $n \geq K$,

$$|\frac{1}{b_n} - \frac{1}{y}| = \frac{|y - b_n|}{|b_n y|} < \frac{\epsilon'}{|b_n||y|} \leq \frac{\epsilon}{M|b_n|} \leq \epsilon.$$

Proof of (e)

- Clearly (e) follows from (d) if we show that $\frac{1}{b_n}$ converges to $\frac{1}{y}$. (Note that here we are assuming that $b_n \neq 0$ for every n and $y \neq 0$.)
- Now we need to estimate,

$$|\frac{1}{b_n} - \frac{1}{y}| = |\frac{y - b_n}{b_n y}| = \frac{|y - b_n|}{|b_n y|}.$$

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Once we prove this claim, for $\epsilon > 0$, take $\epsilon' = \frac{\epsilon |y|}{M}$, and choose $K \in \mathbb{N}$ such that

$$|b_n - y| < \epsilon', \quad \forall n \geq K.$$

- Then for $n \geq K$,

$$|\frac{1}{b_n} - \frac{1}{y}| = \frac{|y - b_n|}{|b_n y|} < \frac{\epsilon'}{|b_n||y|} \leq \frac{\epsilon}{M|b_n|} \leq \epsilon.$$

- This shows that $\frac{1}{b_n}$ converges to $\frac{1}{y}$.

Boundedness

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.

Boundedness

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Proof of claim: Recall that $\lim_{n\to\infty} b_n = y$ and $y \neq 0$.

Boundedness

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Proof of claim: Recall that $\lim_{n\to\infty} b_n = y$ and $y \neq 0$.
- Take $\epsilon = \frac{|y|}{2} > 0$.

Boundedness

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Proof of claim: Recall that $\lim_{n\to\infty} b_n = y$ and $y \neq 0$.
- Take $\epsilon = \frac{|y|}{2} > 0$.
- Now there exists natural number K such that

$$|b_n - y| < \frac{|y|}{2}, \quad \forall n \geq K.$$

Boundedness

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Proof of claim: Recall that $\lim_{n\to\infty} b_n = y$ and $y \neq 0$.
- Take $\epsilon = \frac{|y|}{2} > 0$.
- Now there exists natural number K such that

$$|b_n - y| < \frac{|y|}{2}, \quad \forall n \geq K.$$

- This implies that $|b_n| \geq \frac{|y|}{2}$ for $n \geq K$. (Why?)

Boundedness

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Proof of claim: Recall that $\lim_{n\to\infty} b_n = y$ and $y \neq 0$.
- Take $\epsilon = \frac{|y|}{2} > 0$.
- Now there exists natural number K such that

$$|b_n - y| < \frac{|y|}{2}, \quad \forall n \geq K.$$

- This implies that $|b_n| \geq \frac{|y|}{2}$ for $n \geq K$. (Why?)
- Therefore $\frac{1}{|b_n|} \leq \frac{2}{|y|}$ for $n \geq K$.

Boundedness

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Proof of claim: Recall that $\lim_{n\to\infty} b_n = y$ and $y \neq 0$.
- Take $\epsilon = \frac{|y|}{2} > 0$.
- Now there exists natural number K such that

$$|b_n - y| < \frac{|y|}{2}, \quad \forall n \geq K.$$

- This implies that $|b_n| \geq \frac{|y|}{2}$ for $n \geq K$. (Why?)
- Therefore $\frac{1}{|b_n|} \leq \frac{2}{|y|}$ for $n \geq K$.
- Take

$$M = \max\{\frac{1}{|b_1|}, \frac{1}{|b_2|}, \ldots, \frac{1}{|b_{K-1}|}, \frac{2}{|y|}\}.$$

Boundedness

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Proof of claim: Recall that $\lim_{n\to\infty} b_n = y$ and $y \neq 0$.
- Take $\epsilon = \frac{|y|}{2} > 0$.
- Now there exists natural number K such that

$$|b_n - y| < \frac{|y|}{2}, \quad \forall n \geq K.$$

- This implies that $|b_n| \geq \frac{|y|}{2}$ for $n \geq K$. (Why?)
- Therefore $\frac{1}{|b_n|} \leq \frac{2}{|y|}$ for $n \geq K$.
- Take

$$M = \max\{\frac{1}{|b_1|}, \frac{1}{|b_2|}, \ldots, \frac{1}{|b_{K-1}|}, \frac{2}{|y|}\}.$$

- Note that M is well-defined as $b_n \neq 0$ for every n.
- Now we have $\frac{1}{|b_n|} \leq M$ for every $n \in \mathbb{N}$.

Boundedness

- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Proof of claim: Recall that $\lim_{n\to\infty} b_n = y$ and $y \neq 0$.
- Take $\epsilon = \frac{|y|}{2} > 0$.
- Now there exists natural number K such that

$$|b_n - y| < \frac{|y|}{2}, \quad \forall n \geq K.$$

- This implies that $|b_n| \geq \frac{|y|}{2}$ for $n \geq K$. (Why?)
- Therefore $\frac{1}{|b_n|} \leq \frac{2}{|y|}$ for $n \geq K$.
- Take

$$M = \max\{\frac{1}{|b_1|}, \frac{1}{|b_2|}, \ldots, \frac{1}{|b_{K-1}|}, \frac{2}{|y|}\}.$$

- Note that M is well-defined as $b_n \neq 0$ for every n.
- Now we have $\frac{1}{|b_n|} \leq M$ for every $n \in \mathbb{N}$.
- END OF LECTURE 16.