ANALYSIS -I

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 17. Sequences and order

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$$|a_n - x| < \epsilon, \quad \forall n \geq K.$$
In such a case, $\{a_n\}_{n\in\mathbb{N}}$ is said to converge to x, and x is said to be the limit of $\{a_n\}_{n\in\mathbb{N}}$.
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- We have seen that every convergent sequence is bounded but the converse is not true.

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- (b) $\{a_n + b_n\}_{n\in\mathbb{N}}$ converges to $x + y$.
- (c) For $c, d \in \mathbb{R}$, $\{ca_n + db_n\}_{n\in\mathbb{N}}$ converges to $cx + dy$.
- (d) $\{a_n b_n\}_{n\in\mathbb{N}}$ converges to xy.

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- (c) For $c, d \in \mathbb{R}$, $\{ca_n + db_n\}_{n\in\mathbb{N}}$ converges to $cx + dy$.
- (d) $\{a_n b_n\}_{n\in\mathbb{N}}$ converges to xy.
- (e) If $b_n \neq 0$ for every $n \in \mathbb{N}$ and $y \neq 0$ then $\{\frac{a_n}{b_n}\}_{n\in\mathbb{N}}$ converges to $\frac{x}{y}$.

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- Take $\epsilon = \frac{|x|}{2}$.
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- Clearly this is not possible, as $a_n \geq 0$ and

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- So we have a contradiction. Hence $x < 0$ is not possible.

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- Theorem 17.2: Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences converging to x, y respectively. Suppose $a_n \leq b_n$ for every n. Then $x \leq y$.

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- Also $c_n \geq 0, \forall n$.
- Hence by previous theorem $y - x \geq 0$, or equivalently $x \leq y$.
- Warning: In this Theorem, $a_n < b_n$ for all n does not imply $x < y$. For example, take $a_n = 0$ and $b_n = \frac{1}{n}$ for all n. Then $x = y = 0$ and we don't have $x < y$.

Squeeze theorem

- Theorem 17.3 (Squeeze theorem): Suppose $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ are three sequences satisfying $a_n \leq b_n \leq c_n, \quad \forall n \in \mathbb{N}.$

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- Hence $\{b_n\}_{n\in\mathbb{N}}$ converges to x.

Monotonicity

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- Example 17.5: The sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a decreasing sequence. The sequence $\{n\}_{n\in\mathbb{N}}$ is an increasing sequence.
- Note that an increasing sequence is always bounded below by the first term, that is, $a_1 \leq a_n, \quad \forall n \in \mathbb{N}$ and similarly a decreasing sequence is always bounded above by the first term.

Bounded monotonic sequences

- Theorem 17.6: (i) An increasing sequence $\{a_n\}_{n \in \mathbb{N}}$ is convergent if and only if it is bounded above. In such a case,

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- (iii) A monotonic sequence is convergent if and only if it is bounded.
- Proof: Clearly (iii) follows from (i) and (ii).
- Also (ii) follows from (i), by considering $\{-a_n\}_{n\in\mathbb{N}}$. So it suffices to prove (i).

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- Then by monotonicity of $\{a_n\}_{n\in\mathbb{N}}$ and as x is an upper-bound, we get

$$x - \epsilon < a_K \leq a_n \leq x, \quad \forall n \geq K$$

- In particular,

$$a_n \in (x - \epsilon, x + \epsilon), \quad \forall n \geq K.$$

- This shows that $\{a_n\}_{n\in\mathbb{N}}$ converges to x.

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- Now the result $y = \lim_{n \to \infty} a_n$, is clear from the previous theorem.

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- Inductively, one can show that

$$a = a_1 \leq a_2 \leq \cdots a_n \leq b_n \leq \cdots \leq b_2 \leq b_1 = b.$$

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- This value is known as arithmetic-geometric mean of a and b. 7.

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- END OF LECTURE 17.