ANALYSIS -I

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Lecture 18. Bolzano-Weierstrass theorem

- We recall a few notions from the previous lecture.
- Definition 17.4: A sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers is said to be increasing (or non-decreasing) if

$$a_n \leq a_{n+1}, \quad \forall n \in \mathbb{N}.$$

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- Example 17.5: The sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a decreasing sequence. The sequence $\{n\}_{n\in\mathbb{N}}$ is an increasing sequence.
- Note that an increasing sequence is always bounded below by the first term, that is, $a_1 \leq a_n, \quad \forall n \in \mathbb{N}$ and similarly a decreasing sequence is always bounded above by the first term.

Bounded monotonic sequences

- Theorem 17.6: (i) An increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ is convergent if and only if it is bounded above. In such a case,

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- (iii) A monotonic sequence is convergent if and only if it is bounded.

Subsequences

- Definition 18.1: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Let

$$n_1 < n_2 < n_3 < \cdots$$

be a strictly increasing sequence of natural numbers. Then $\{a_{n_k}\}_{k\in\mathbb{N}}$ or equivalently,

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- Example 18.2: Let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence defined by $a_n = \frac{1}{n}$. Taking $n_k = k^2$, we get get the subsequence
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- It is the sequence $\{\frac{1}{k^2}\}_{k\in\mathbb{N}}$. Taking $m_k = 2^k$, we get a new subsequence $\{a_{m_k}\}_{k\in\mathbb{N}}$, which is,
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- Such subsequences are known as tails of the given sequence.

Subsequences of convergent sequences

- Theorem 18.4: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers converging to some $x \in \mathbb{R}$. Then every subsequence of $\{a_n\}_{n\in\mathbb{N}}$ converges to x. In particular, every tail of this sequence converges to x.

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- Hence $\{a_{n_k}\}_{k\in\mathbb{N}}$ converges to x.

Limit points

- Definition 18.5: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then $y \in \mathbb{R}$ is said to be limit point of $\{a_n\}_{n\in\mathbb{N}}$, if it has a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ converging to y.

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- Example 18.6: Define a sequence $\{c_n\}_{n\in\mathbb{N}}$ by

$$c_n = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even} \end{cases}$$

Then clearly $2, 3$ are limit points of this sequence. It is an exercise to show that there are no other limit points.

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- Can a sequence have infinitely many limit points?

Examples

- Example 18.7: Consider the enumeration of $\mathbb{N} \times \mathbb{N}$ as

$$(1,1),(1,2),(2,1),(3,1),(2,2),(1,3),(1,4),(2,3),(3,2),(4,1),(5,1$$

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$$2, 3, 3, 4, 4, 4, \ldots.$$

- Now consider the function $(m,n) \mapsto \frac{1}{m} + \frac{1}{n}$. Applying this function on the enumeration above we get a sequence of real numbers as:

$$\frac{1}{1} + \frac{1}{1}, \quad \frac{1}{2} + \frac{1}{1}, \quad \frac{1}{2} + \frac{1}{1}, \quad \frac{1}{3} + \frac{1}{1}, \quad \frac{1}{1} + \frac{1}{4}, \quad \ldots.$$

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- It is an exercise to show that the set of limit points of this sequence is given by

$$\{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{0\}.$$

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- Call a natural number m as a peak for $\{a_n\}_{n\in\mathbb{N}}$ if $a_m \geq a_n$ for all $n \geq m$. In other words m is a peak if a_m is an upper bound for $\{a_m, a_{m+1}, a_{m+2}, \ldots\}$.

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- It is possible that P is the empty set.

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- Continuing this way, after choosing n_k, we can choose n_{k+1}, where $n_{k+1} > n_k$ and $a_{n_{k+1}} > a_{n_k}$.
- In other words, we have an increasing subsequence in:

$$a_{n_1} < a_{n_2} < a_{n_3} < \cdots$$

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- Obviously, this monotonic subsequence is bounded as the original sequence is bounded.
- As every bounded monotonic sequence is convergent, this subsequence is convergent. This completes the proof.

Sequential Compactness

- Theorem 18.11: Suppose $[a, b]$ is an interval and $\{c_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers with $c_n \in [a, b]$. Then $\{c_n\}_{n\in\mathbb{N}}$ has a convergent subsequence and any such subsequence converges to a point in $[a, b]$.

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- Note that the same property does not hold for intervals like (a, b) as the limit may not be an element of the interval.

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- END OF LECTURE 18.