ANALYSIS -I

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

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- As every bounded monotonic sequence is convergent, this subsequence is convergent. This completes the proof.

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- Definition 19.1: A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to be Cauchy if for every $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that

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- We may write $|a_m - a_n| < \epsilon$ equivalently as $a_m \in (a_n - \epsilon, a_n + \epsilon)$ or as $(a_m - a_n) \in (-\epsilon, +\epsilon)$.

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- Using convergence of $\{a_{n_k}\}_{k\in\mathbb{N}}$, choose K_2 such that

$$|a_{n_k} - x| < \frac{\epsilon}{2}, \quad \forall k \geq K_2.$$

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- Take $K = \max\{K_1, n_{K_2}\}$. Note that $n_K \geq K \geq K_1$ and $K \geq K_2$.

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- The set of real numbers is complete due to least upper bound axiom, where as $\mathbb{Q}$ is not complete. For this reason the least upper bound axiom is also known as completeness axiom.
- There is a way of completing every metric space and if we complete $\mathbb{Q}$ by this procedure we get the set of real numbers $\mathbb{R}$. This is one way of constructing $\mathbb{R}$.

Infinite series

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- It is a natural question as to when $\sum_{j=1}^{\infty} a_j$ or

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  is meaningful.
- Definition 19.5: Suppose $a_1, a_2, \ldots$ are real numbers. Take $s_n = \sum_{j=1}^{n} a_j$. Here $\{s_n\}_{n\in\mathbb{N}}$ are known as partial sums of the series. If $\lim_{n\to\infty} s_n$ exists then the series, $\sum_{j=1}^{\infty} a_j$ is said to converge and

$$\sum_{j=1}^{\infty} a_j := \lim_{n\to\infty} s_n.$$

  If $\lim_{n\to\infty} s_n$ does not exist, the series $\sum_{j=1}^{\infty} a_j$ is said to diverge.

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- Now

$$\begin{aligned} s_n &:= \sum_{j=1}^{n} \frac{1}{2^j} \\ &= \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \\ &= \frac{1}{2}[1 + \frac{1}{2} + \cdots + (\frac{1}{2})^{(n-1)}] \\ &= \frac{1}{2} . \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} \\ &= 1 - \frac{1}{2^n} \end{aligned}$$

Continuation

- Using Bernoulli's inequality, we have seen that $\frac{1}{2^n} < \frac{1}{n+1}$ and hence $\lim_{n\to\infty} \frac{1}{2^n} = 0$. Hence $\lim_{n\to\infty} s_n = 1$.

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- Similarly, one can show that for any $|r| < 1$, $\lim_{n\to\infty} r^{n-1} = 0$ and

$$1 + r + r^2 + \cdots = \frac{1}{1-r}$$

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- Equivalently, $|a_n| < \epsilon$ for $n \geq K + 1$. Hence $\{a_n\}_{n\in\mathbb{N}}$ converges to 0.
- The converse is not true is seen by considering the 'Harmonic series' :
- $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges as the corresponding partial sums are unbounded.

Alternating sum

- Theorem 19.8: A series $\sum_{j=1}^{\infty} a_j$, where $a_j = (-1)^{j+1} b_j$, with a decreasing sequence $\{b_j\}_{j\in\mathbb{N}}$ of positive real numbers is convergent if and only if $\lim_{n\to\infty} b_n = 0$.

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- Now suppose $\lim_{n\to\infty} b_n = 0$.
- Consider the partial sums

$$s_n = \sum_{j=1}^{n} a_j = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{n+1} b_n.$$

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- We have, $s_{2k+2} = s_{2k} + b_{2k+1} - b_{2k+2}$.

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- Similarly $\{s_{2k-1}\}_{k\in\mathbb{N}}$ is bounded below by $s_2 = b_1 - b_2$.
- That is,

$$b_1 - b_2 = s_2 \leq s_4 \leq \cdots \leq s_{2k} \leq s_{2k-1} \leq \cdots s_3 \leq s_1 = b_1$$

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- END OF LECTURE 19.