ANALYSIS -I

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Lecture 2: Set theory and Russell's paradox

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- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$-the set of integers.

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- This is a requirement so that we do not have any confusion. Still the definition is only an informal one.

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- Let us see some more paradoxes of similar type.

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- What about the adjective 'HETEROLOGICAL? We again face a problem.

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- There is a problem with the Second Catalogue. Should it list itself or not?

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