# ANALYSIS -I

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- Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers.

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- By previous theorem there exists a monotonic subsequence of $\{a_n\}_{n\in\mathbb{N}}$.
- Obviously, this monotonic subsequence is bounded as the original sequence is bounded.
- As every bounded monotonic sequence is convergent, this subsequence is convergent. This completes the proof.

# Limit points

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- Definition 18.5: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then $y \in \mathbb{R}$ is said to be limit point of $\{a_n\}_{n\in\mathbb{N}}$, if it has a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ converging to $y$.
- We would like to understand the structure of limit points better. The following theorem is easy to prove.

# Terms around a limit point

- Theorem 20.1: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then $y \in \mathbb{R}$ is a limit point of the sequence $\{a_n\}_{n\in\mathbb{N}}$ if and only if the set
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- In other words, there are infinitely many terms of the sequence in $(y-\epsilon, y+\epsilon)$ for every $\epsilon > 0$.
- Proof: Suppose for $k \in \mathbb{N}$,

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is infinite for every $k$. Then it is easy to choose a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ such that

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- The converse is also easy to see.

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- In conclusion, $\{b_n\}$ is a bounded decreasing sequence. Hence $\lim_{n\to\infty} b_n$ exists.

# Continuation

- Definition 20.2: For any bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, the $\lim_{n\to\infty} b_n$ defined as above is known as the limit superior or limsup of the bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, and we write:

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- In other words, the 'limsup' is the limit of supremums of tails of the sequence.

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- Definition 20.3: For any bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, the $\lim_{n\to\infty} c_n$ defined as above is known as the limit inferior or liminf of the bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, and we write:

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- A bounded sequence may not be convergent and so it may not have a limit. But it always has liminf and limsup.

# Examples

- Example 20.4: Consider the sequence $\{a_n\}$ where,

$$a_n = \begin{cases} 5 & \text{if } n = 3k+1, k \in \mathbb{N} \bigcup \{0\} \\ 6 & \text{if } n = 3k+2, k \in \mathbb{N} \bigcup \{0\} \\ 7 & \text{if } n = 3k, k \in \mathbb{N}. \end{cases}$$

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- Then $b_n = 3$ for every $n$ and $c_n = 0$ for every $n$.
- In particular, it is not immediate that limsup and liminf are limit points of the sequence.

# A Characterization

- Theorem 20.6: Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers and suppose $z = \limsup_{n\to\infty} a_n$. Then for every $\epsilon > 0$, the set

$$S_+(z,\epsilon) = \{n : a_n > z + \epsilon\} \text{ is finite.} \quad (*)$$

and the set

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- Conversely if $v \in \mathbb{R}$ satisfies $(*), (**)$ for every $\epsilon > 0$, with $z$ replaced by $v$, then $v = z$.
- Proof: Suppose $z = \limsup_{n\to\infty} a_n$.
- Fix $\epsilon > 0$. Take $b_n = \sup\{a_m : m \geq n\}$. By the definition of limsup, $z = \lim_{n\to\infty} b_n$.

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$$S_-(z,\epsilon) = \{n : a_n > z - \epsilon\} \text{ is infinite.} \quad (**)$$

- Conversely if $v \in \mathbb{R}$ satisfies $(*), (**)$ for every $\epsilon > 0$, with $z$ replaced by $v$, then $v = z$.
- Proof: Suppose $z = \limsup_{n\to\infty} a_n$.
- Fix $\epsilon > 0$. Take $b_n = \sup\{a_m : m \geq n\}$. By the definition of limsup, $z = \lim_{n\to\infty} b_n$.
- Hence there exists $K \in \mathbb{N}$ such that

$$b_n \in (z-\epsilon, z+\epsilon), \quad \forall n \geq K.$$

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- Now it is clear that $S_-(z, \epsilon)$ is infinite.

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- This allows us to choose a subsequence $\{a_{n_r}\}_{r\in\mathbb{N}}$, where $v - \frac{1}{r} < a_{n_r}$. Then $v - \frac{1}{r} < b_{n_r}$, and hence on taking limit as $r \to \infty$, $v \leq \lim_{r\to\infty} b_{n_r} = z$. That is, $v \leq z$. Combining the two statements we have $v = z$.

# limsup as a limit point

- Theorem 20.7: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence of real numbers. Then $\limsup_{n\to\infty} a_n$ is a limit point of $\{a_n\}_{n\in\mathbb{N}}$ and if $y$ is any limit point of $\{a_n\}_{n\in\mathbb{N}}$, then $y \leq \limsup_{n\to\infty} a_n$.

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