ANALYSIS -I

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is infinite for every $\epsilon > 0$.

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  is infinite for every $\epsilon > 0$.
- In other words, there are infinitely many terms of the sequence in $(y-\epsilon, y+\epsilon)$ for every $\epsilon > 0$.

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- In conclusion, $\{b_n\}$ is a bounded decreasing sequence. Hence $\lim_{n\to\infty} b_n$ exists.

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- Definition 20.2: For any bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, the $\lim_{n\to\infty} b_n$ defined as above is known as the limit superior or limsup of the bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, and we write:

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- A bounded sequence may not be convergent and so it may not have a limit. But it always has liminf and limsup.

A Characterization

- Theorem 20.6: Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers and suppose $z = \limsup_{n\to\infty} a_n$. Then for every $\epsilon > 0$, the set

$$S_+(z,\epsilon) = \{n : a_n > z + \epsilon\} \text{ is finite.} \quad (*)$$

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- Conversely if $v \in \mathbb{R}$ satisfies $(*), (**)$ for every $\epsilon > 0$, with z replaced by v, then $v = z$.

Limit superior as a limit point

- Theorem 20.7: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence of real numbers. Then $\limsup_{n\to\infty} a_n$ is a limit point of $\{a_n\}_{n\in\mathbb{N}}$ and if y is any limit point of $\{a_n\}_{n\in\mathbb{N}}$, then $y \leq \limsup_{n\to\infty} a_n$.

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- Hence z is a limit point of $\{a_n\}_{n\in\mathbb{N}}$.
- The fact that z is the largest limit point is also clear from the characterization for if $z < v$, then taking $\epsilon = \frac{v-z}{2}$, $(v-\epsilon, v+\epsilon) \subseteq S_+(z,\epsilon)$ has finitely many terms of the sequence.

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- Similarly liminf is the smallest limit point of a bounded sequence.

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- Consequently, the set of limit points of a bounded sequence $\{a_n\}_{n\in\mathbb{N}}$ is a subset of $[w, z]$ where $w = \liminf_{n\to\infty} a_n$ and $z = \limsup_{n\to\infty} a_n$.

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- If liminf and limsup are equal. Then as we have

$$c_n \leq a_n \leq b_n, \quad \forall n \in \mathbb{N}$$

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- This shows that when we do not know whether a sequence is convergent or not, we may try to compute its liminf and limsup and see whether they are equal or not.

Properly divergent sequences

- Definition 21.3: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then it is said to properly diverge to $+\infty$ if for every $M \in \mathbb{R}$ there exists $K \in \mathbb{N}$ such that

$$a_n \geq M, \quad \forall n \geq K.$$

This is written as:

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- A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to properly diverge to $-\infty$, if for every $M \in \mathbb{R}$ there exists $K \in \mathbb{N}$ such that $a_n < M$ for all $n \geq K$. This is expressed as: $\lim_{n\to\infty} a_n = -\infty$.

Properly divergent sequences

- Definition 21.3: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then it is said to properly diverge to $+\infty$ if for every $M \in \mathbb{R}$ there exists $K \in \mathbb{N}$ such that
$$a_n \geq M, \quad \forall n \geq K.$$
This is written as:
$$\lim_{n\to\infty} a_n = +\infty.$$
or as
$$\lim_{n\to\infty} a_n = \infty.$$
- A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to properly diverge to $-\infty$, if for every $M \in \mathbb{R}$ there exists $K \in \mathbb{N}$ such that $a_n < M$ for all $n \geq K$. This is expressed as: $\lim_{n\to\infty} a_n = -\infty$.
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This is written as:

$$\lim_{n\to\infty} a_n = +\infty.$$

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- A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to properly diverge to $-\infty$, if for every $M \in \mathbb{R}$ there exists $K \in \mathbb{N}$ such that $a_n < M$ for all $n \geq K$. This is expressed as: $\lim_{n\to\infty} a_n = -\infty$.
- A sequence is said to properly diverge if it properly diverges to $+\infty$ or $-\infty$.
- Here $+\infty$ and $-\infty$ are not real numbers. It is just convenient notation.

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- However, it should be kept in mind that such sequences are not convergent sequences in a proper sense as $+\infty$ and $-\infty$ are not real numbers.
- Example 21.4: Define:

$$a_n = n^2, \quad \forall n \in \mathbb{N}.$$

$$b_n = \begin{cases} 5 & \text{if } n \text{ is odd.} \\ n & \text{if } n \text{ is even.} \end{cases}$$

$$c_n = \begin{cases} 5 & \text{if } n \text{ is odd.} \\ 6 & \text{if } n \text{ is even.} \end{cases}$$

Here $\{a_n\}_{n\in\mathbb{N}}$ is properly divergent to $+\infty$, $\{b_n\}_{n\in\mathbb{N}}$ is unbounded and divergent but it is not properly divergent, $\{c_n\}_{n\in\mathbb{N}}$ is bounded and divergent but not properly divergent.

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- Proof: Without loss of generality, take $M > 0$. There exists $K_1 \in \mathbb{N}$ such that $a_n \geq M$ for $n \geq K_1$.

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- Proofs of other claims are left out as exercises.

Some more properties

- Theorem 21.6: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequences of real numbers properly diverging to $+\infty$ and let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence converging to some real number x.

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- (ii) If $x > 0$, $\{a_n b_n\}_{n\in\mathbb{N}}$ properly diverges to $+\infty$. If $x < 0$, $\{a_n b_n\}_{n\in\mathbb{N}}$ properly diverges to $-\infty$.

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- (iii) If $x > 0$ and $b_n \neq 0$ for every n, then $\{\frac{a_n}{b_n}\}$ properly diverges to ∞. If $x < 0$ and $b_n \neq 0$ for every n, then $\{\frac{a_n}{b_n}\}$ properly diverges to $-\infty$.

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- (iv) If $a_n \neq 0$ for every n, then $\{\frac{b_n}{a_n}\}$ converges to 0.

Continuation

- If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ properly diverge to $+\infty$, $\{a_n - b_n\}_{n\to\infty}$ may not converge. Similarly $\{\frac{a_n}{b_n}\}_{n\in\mathbb{N}}$ need not converge.

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- Give examples to illustrate such phenomenon.

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- END OF LECTURE 21