ANALYSIS -I

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Lecture 22. Continuous functions

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$$|f(x) - f(c)| \leq |x^2 - c^2| = |x - c||x + c| < \frac{\epsilon}{2}.(|x| + |c|) \leq \frac{\epsilon}{2}.2 = \epsilon.$$

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- Therefore f is continuous at c.

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- Example 22.3: Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 5 & \text{if } x = 1. \end{cases}$$

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- For any $\epsilon < 5$, there is no $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon, \quad \forall x \in (c - \delta, c + \delta) \bigcap [0, 1].$$

Sequential form of continuity

- Theorem 22.4: Let $A \subseteq \mathbb{R}$ and let $c \in A$. Then a function $f : A \to \mathbb{R}$ is continuous at c, if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in A, converging to c,

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- This shows that $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(c)$.

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- This completes the proof

More Examples

- Example 22.5: Suppose $A = \{1\} \bigcup [2, 3]$ and $g : A \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} 0 & \text{if} \quad x = 1; \\ 7 & \text{if} \quad x \in [2, 3]. \end{cases}$$

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- Remark 22.6: Suppose $A \subset \mathbb{R}$ and $c \in A$ is isolated in A. Then every function $f : A \to \mathbb{R}$ is continuous at c.

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