ANALYSIS -I

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Lecture 23. Algebraic operations of Continuous functions

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Lecture 23. Algebraic operations of Continuous functions

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- Definition 22.1: Let $A \subseteq \mathbb{R}$ and let $c \in A$. Then a function $f : A \to \mathbb{R}$ is said to be continuous at c, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon, \quad \forall x \in (c - \delta, c + \delta) \bigcap A.$$

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- Definition 22.7: Let $A \subseteq \mathbb{R}$. Then a function $f : A \to \mathbb{R}$ is said to be continuous if f is continuous at every $c \in A$.

Linear combinations, products, ratios

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First Proof

- Proof. (i) For $\epsilon > 0$, using continuity of f at c, choose $\delta_1 > 0$, such that

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- Similarly using continuity of g at c, choose $\delta_2 > 0$ such that

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- Now take $\delta = \min\{\delta_1, \delta_2\}$. Then for $x \in (c - \delta, c + \delta) \bigcap A$, we get
$$|f(x)+g(x)-f(c)-g(c)| \leq |f(x)-f(c)|+|g(x)-g(c)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

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- Therefore $f + g$ is continuous at c.
- It is easy to see that if f is continuous at c, af is continuous at c. Similarly bg is continuous at c. Combining with the previous result, $af + bg$ is continuous at c.

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- Hence, $\{af(x_n) + bg(x_n)\}_{n\in\mathbb{N}}$ converges to $af(c) + bg(c)$.

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- Similarly, $\{f(x_n)g(x_n)\}$ converges to $f(c)g(c)$ and if $g(x) \neq 0$ for every x, $\{\frac{f(x_n)}{g(x_n)}\}$ converges to $\frac{f(c)}{g(c)}$.

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- Hence fg and $\frac{f}{g}$ are continuous. This completes the proof.

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  is continuous.
- Proof: This is clear from the previous theorem and the definition of continuous functions.

Restrictions of continuous functions

- Theorem 23.3: Let $A \subseteq \mathbb{R}$ and let B be a subset of A and let $c \in B$. Suppose $f : A \to \mathbb{R}$ is a function continuous at c. Then $g : B \to \mathbb{R}$ defined by

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  is continuous at c. If f is continuous, then g is continuous.

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- Proof: This is obvious from the definition of continuity.
- Notation: The function g of this theorem is called the restriction of f to B and is denoted by $f|_B$.

Continuity of polynomials

- Theorem 23.4: Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial defined by

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad \forall x \in \mathbb{R},$$

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- Proof: It is easy to see that the constant function

$$p_0(x) = a_0, \quad x \in \mathbb{R}$$

and the identity function,

$$p_1(x) = x, \quad x \in \mathbb{R}$$

are continuous. Now by (ii) of Theorem 23.2, and mathematical induction, the polynomials

$$p_k(x) = x^k, \quad \forall x \in \mathbb{R}$$

$k \in \mathbb{N}$, are continuous. The proof is complete by a simple application of (i) of Theorem 23.2.

- Corollary 23.5: For any non-empty subset B of $\mathbb{R}$ and any real polynomial p, $p|_B$, defined by

$$p|_B(x) = p(x), \quad x \in B$$

is continuous.

Rational functions

- Corollary 23.5: For any non-empty subset B of $\mathbb{R}$ and any real polynomial p, $p|_B$, defined by

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- If q is another polynomial such that $q(x) \neq 0$ for $x \in B$, then $\frac{p|_B}{q|_B}$ is a continuous function on B.

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- Such functions are known as rational functions.
- Example 23.6: The function $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}, \quad \forall x \in \mathbb{R}\backslash\{0\}$ is continuous.

Composition of continuous functions

- Theorem 23.7: Let A, B be subsets of $\mathbb{R}$ and $c \in A$. Suppose f, g are real valued functions on A, B respectively and $f(A) \subseteq B$. Suppose f is continuous at c and g is continuous at $f(c)$. Then $h = g \circ f$ is continuous at c.

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- As $f(A) \subseteq B$, $\{f(x_n)\}$ is a sequence in B.
- Now as g is continuous at $f(c)$, $\{g(f(x_n)\}$ converges to $g(f(c))$.

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- Proof: Suppose $\{x_n\}_{n \in \mathbb{N}}$ in A converges to c. Then as f is continuous, $\{f(x_n)\}$ converges to $f(c)$.
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- In other words $\{h(x_n)\}$ converges to $h(c)$. This proves that h is continuous at c.
- Exercise 23.8: Prove the previous theorem directly using the definition of continuity.

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- Theorem 23.9: Let A, B be subsets of $\mathbb{R}$. Suppose f, g are continuous real valued functions on A, B respectively and $f(A) \subseteq B$. Then $h = g \circ f$ is a continuous function.

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- Theorem 23.9: Let A, B be subsets of $\mathbb{R}$. Suppose f, g are continuous real valued functions on A, B respectively and $f(A) \subseteq B$. Then $h = g \circ f$ is a continuous function.
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- Example 23.10 (Dirichlet function): Define $d : \mathbb{R} \to \mathbb{R}$ by

$$d(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

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- Then d is discontinuous at every $x \in \mathbb{R}$.
- Example 23.11: Define $g : [1, 2] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational;} \\ & \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \quad p, q \in \mathbb{N} \\ & \quad p, q \quad \text{relatively prime.} \end{cases}$$

Then g is continuous at irrational points in $[1, 2]$, but is discontinuous at rational points in $[1, 2]$.

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- END OF LECTURE 23.