ANALYSIS -I

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Lecture 24. Continuous functions on intervals

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- Definition 22.1: Let $A \subseteq \mathbb{R}$ and let $c \in A$. Then a function $f : A \to \mathbb{R}$ is said to be continuous at c, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

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- Definition 22.7: Let $A \subseteq \mathbb{R}$. Then a function $f : A \to \mathbb{R}$ is said to be continuous if f is continuous at every $c \in A$.

Boundedness of functions

- Definition 24.1: Let A be a non-empty set and let $f : A \to \mathbb{R}$ be a function. Then f is said to be bounded if

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- Similarly, $\inf(f)$ is said to be a minimum if there exists $x_1 \in A$ such that $f(x_1) = \inf(f)$.

Examples

- Example 24.2: Let $f : [0,1) \to \mathbb{R}$ be the function $f(x) = x, \quad \forall x \in [0,1)$. Then f is bounded with bound 1. $\sup(f)$ is not a maximum. However, inf is a minimum with $\inf(f) = f(0)$.

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- Example 24.3: Let $g : (0, 1) \to \mathbb{R}$ be the function $g(x) = \frac{1}{x}, \quad x \in (0, 1)$. Then f is continuous but not bounded.

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- Now $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $[a, b]$.
- Then by Bolzano-Weierstrass theorem there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$.

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- This is a contradiction and this completes the proof.
- We have already seen that continuous functions on open intervals need not be bounded. Also examples, such as $f(x) = x$, show that continuous functions on $\mathbb{R}$ need not be bounded.

Maximum and minimum

- Theorem 24.5: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists c, d in $[a, b]$ such that

$$f(c) = \sup\{f(x) : x \in [a, b]\};$$

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- Now for $n \in \mathbb{N}$, as $M - \frac{1}{n}$ is not an upper bound of this set, there exists $x_n \in [a, b]$ such that

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- By squeeze theorem,

$$\lim_{n\to\infty} f(x_n) = M.$$

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