ANALYSIS -I

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Lecture 25. Intermediate value theorem

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- Theorem 24.5: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists c, d in $[a, b]$ such that

$$f(c) = \sup\{f(x) : x \in [a, b]\};$$

$$f(d) = \inf\{f(x) : x \in [a, b]\}.$$

Existence of roots: Bisection method

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- Continuing this way, after choosing $I_n = [a_n, b_n]$, either $f(\frac{a_n+b_n}{2}) = 0$ or we have $I_{n+1} = [a_{n+1}, b_{n+1}]$, in such a way that $I_n \supset I_{n+1}$ with $(b_{n+1} - a_{n+1}) = \frac{1}{2}(b_n - a_n)$.

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- Suppose $\{c\} = \bigcap_{n\in\mathbb{N}} I_n$.
- We clearly have $\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$.

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- Remark: Any point x such that $f(x) = 0$ is some times, especially when f is a polynomial, is called a root of f or zero of f.
- In this proof we have seen a way of locating the root by successively bisecting the interval.

Intermediate value theorem

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- If $f(a) > z > f(b)$, consider g defined by

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and similar proof works.

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- Clearly, p is continuous and is unbounded.
- Therefore, we can get a b such that $t < p(b)$. (Exercise: We may take $b = t + 1$.)

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- In other words if $0 < c < d$, we have $c^n < d^n$ and so we can't have $c^n = d^n$.. This shows the uniqueness of positive n^{th} root of t.

Roots of polynomials

- Example 25.4: Consider the polynomial $p(x) = x^3 - 2x^2 - 1$. Show that there exists a real number λ such that $0 < \lambda < 3$ and $p(\lambda) = 0$.

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- Exercise 25.5: Suppose p is an odd degree real polynomial. Show that there exists a real number λ such that $p(\lambda) = 0$.

Continuous image of an interval

- Theorem 25.6: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then
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- If $s < t$, and $s < z < t$, we want to show that there exists $e \in [a, b]$ such that $f(e) = z$.
- But this is clear from the inter mediate value theorem as there exist c, d in $[a, b]$ such that $f(c) = s$ and $f(d) = t$.

Continuous images of arbitrary intervals

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Continuous images of arbitrary intervals

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- Theorem 25.7: Suppose $I \subseteq \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is a continuous function. Then $f(I)$ is an interval.
- Recall that intervals are sets of the form $\{a\}, [a, b], [a, b), (a, b], [a, \infty), (a, \infty), (-\infty, b], (-\infty, b), (-\infty, \infty)$, with $a, b \in \mathbb{R}, a < b$.

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- Exercise 25.8: Show that a non-empty subset S of $\mathbb{R}$ is an interval if and only if $x, y \in S$ with $x < y$ implies $[x, y] \subseteq S$.

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- Exercise 25.8: Show that a non-empty subset S of $\mathbb{R}$ is an interval if and only if $x, y \in S$ with $x < y$ implies $[x, y] \subseteq S$.
- Now the proof of Theorem 25.7 follows easily from the inter mediate value theorem.

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- Sketch of proof:
- We model the equator by a circle, or by the interval $[0, 1]$, where we identify the points 0 and 1.

Continuation

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- This means that $f(c) - f(c + \frac{1}{2}) = 0$ or

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- Then $g(\frac{1}{2}) = -g(0)$. In other words $g(0)$ and $g(\frac{1}{2})$ have opposite signs.
- If $g(0) = 0$, 0 and $\frac{1}{2}$ are antipodal points with equal temperature. So we may assume $g(0) \neq 0$.
- Assume that g is continuous. Then by intermediate value theorem there exists $c \in [0, \frac{1}{2}]$ such that $g(c) = 0$.
- This means that $f(c) - f(c + \frac{1}{2}) = 0$ or

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- END OF LECTURE 25.