ANALYSIS -I

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 26. Uniform continuity and monotonicity

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- Definition 22.1: Let $A \subseteq \mathbb{R}$ and let $c \in A$. Then a function $f : A \to \mathbb{R}$ is said to be continuous at c, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

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- Definition 22.7: Let $A \subseteq \mathbb{R}$. Then a function $f : A \to \mathbb{R}$ is said to be continuous if f is continuous at every $c \in A$.

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- Suppose $f : A \to \mathbb{R}$ is continuous at every y in A. Then we have for every $\epsilon > 0$, there exists δ, depending on y, such that

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- It is important here that the δ here depends only on ϵ and not on x or y.

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- Exercise 26.4: Show that $f : (0,1) \to (0,1)$ defined by
$$f(x) = \frac{1}{x}, \ \ \forall x \in (0,1),$$
is not uniformly continuous.

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  holds.
- In particular, this inequality does not hold for $\delta = \frac{1}{n}$ for every $n \in \mathbb{N}$.
- This means that there exist x_n, y_n in $[a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ and

$$|f(x_n) - f(y_n)| \geq \epsilon_0.$$

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- (i) $\{z_k\}_{k\in\mathbb{N}}$ converges to z.
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$$z_k - \frac{1}{k} \leq w_k \leq z_k + \frac{1}{k}, \quad \forall k \in \mathbb{N}.$$

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- This contradicts, (iii), as we can choose, K_1 such that

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- Similarly there exists K_2 such that,

$$|f(w_k) - f(z)| < \frac{\epsilon_0}{2}, \quad \forall k \geq K_2.$$

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- Take $K = \max\{K_1, K_2\}$. Then by triangle inequality we have,

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- Therefore f is uniformly continuous.

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Continuous bijections

- Theorem 26.7: Let a, b, a', b' be real numbers with $a < b$ and $a' < b'$. If $f : [a, b] \to [a', b']$ is a continuous bijection then either f is strictly increasing with $f(a) = a'$ and $f(b) = b'$ or f is strictly decreasing with $f(a) = b'$ and $f(b) = a'$

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- Proof: We know that any continuous function f on $[a, b]$ maps $[a, b]$ onto $[s, t]$ where

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- Also as the infimum and supremum are attained there exist, c, d in $[a, b]$ such that $f(c) = s = a'$ and $f(d) = t = b'$.
- We claim that if $c < d$, then f is strictly increasing. By intermediate value theorem, $f([c, d]) = [a', b']$. Now the bijectivity of f forces $c = a$ and $d = b$, so that $f(a) = a'$ and $f(b) = b'$.

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- Since $f(a) = a'$ and $f(x) > f(y)$, $x = a$ is not possible.

Continuation

- If f is not strictly increasing, there exist x, y in $[a, b]$ such that $x < y$ and $f(x) > f(y)$ (Since f is injective $f(x) = f(y)$ is ruled out.)
- Since $f(a) = a'$ and $f(x) > f(y)$, $x = a$ is not possible.
- So we have $a < x < y \leq b$ and $f(a) = a'$, and $f(x) > f(y) > a'$

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- On applying intermediate value theorem to $f|_{[a,x]}$ there must be some $z \in [a, x]$ such that $f(z) = f(y)$. This contradicts injectivity of f.

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- Therefore if $c < d$, then f is strictly increasing and $f(a) = a', f(b) = b'$.

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- Finally $c = d$ is not possible as f can't be a constant function due to injectivity of f. $\blacksquare$

Continuation

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Continuation

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- END OF LECTURE 26.