ANALYSIS -I

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Lecture 27. Limits to cluster points

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$$(c-\delta, c+\delta) \bigcap A \backslash \{c\} \neq \emptyset.$$

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- Note that we are excluding c from these sequences.

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- Definition 27.4: Let c be a cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then f is said to have a limit at c if there exists $z \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

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- Remark: It should be clear that if f has a limit at c, then it is unique.
- Notation: If z is the limit of f at c, we write

$$\lim_{x \to c} f(x) = z.$$

Sequential version

- Proposition 27.5: Let c be a cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then z is limit of f at c if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A\backslash\{c\}$ converging to c, $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to z.

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- Then for $n \geq K$, $x_n \in (c-\delta, c+\delta)\bigcap(A\backslash\{c\})$ and hence $|f(x_n) - z| < \epsilon, \quad \forall n \geq K.$

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- Then for $n \geq K$, $x_n \in (c - \delta, c + \delta) \bigcap (A\backslash\{c\})$ and hence $|f(x_n) - z| < \epsilon, \quad \forall n \geq K$.
- Therefore $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(c)$.

Continuation

- Now suppose z is not a limit of f at c. Then there exists $\epsilon_0 > 0$ such that for no $\delta > 0$

$$|f(x) - z| < \epsilon_0, \quad \forall x \in (c - \delta, c + \delta) \bigcap (A \backslash \{c\})$$

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- Clearly then $\{x_n\}_{n \in \mathbb{N}}$ converges to c, but $\{f(x_n)\}$ does not converge to z. $\blacksquare$.

Example

- Example 27.6: Define $h : [0,2)\bigcup(2,3] \to \mathbb{R}$ by

$$h(x) = \begin{cases} 2x & \text{if } x \in [0,2) \\ \frac{(x^3-2x^2)}{x-2} & \text{if } x \in (2,3] \end{cases}$$

extends to a continuous function $\tilde{h}$ on $[0,3]$ by taking $\tilde{h}(x) = h(x)$ for $x \in [0,2)\bigcup(2,3]$ and $\tilde{h}(2) = 4$.

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- Remark: Suppose c is a cluster point of a set $A \subseteq \mathbb{R}$ and $f; A \to \mathbb{R}$ is a function. Suppose $\lim_{x\to c} f(x) = z$, then $\tilde{f} : A\bigcup\{c\} \to \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A\backslash\{c\} \\ z & \text{if } x = c \end{cases}$$

  is continuous at c.

Left and right hand cluster points

- Definition 27.7: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is said to be a right cluster point of A if for every $\delta > 0$

$$(c, c+\delta) \bigcap A \neq \emptyset.$$

Similarly c is said to be a left cluster point of A if for every $\delta > 0$

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- Proof. Exercise.

Left and right hand limits

- Definition 27.9: Let c be a right cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then f is said to have a right hand limit at c if there exists $z \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

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- Observe that,

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iff for every decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in A converging to c, $\{f(x_n)\}$ converges to z.

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- Some texts may have the notation: $\lim_{x \downarrow c} f(x) = z$.

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Monotonic functions

- Theorem 27.11: Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a function. Suppose f is increasing then the following hold.

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- (iii) For every $c \in (a, b)$

$$\lim_{x \to c-} f(x) \leq f(c) \leq \lim_{x \to c+} f(x).$$

Therefore f is continuous at c if and only if

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- (vi) For every $c \in (a, b)$

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- As f is increasing, $z - \epsilon < f(d) \leq f(x) \leq z$ for $d \leq x < c$.

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- Consider any $\epsilon > 0$. Since $z - \epsilon$ is less than the supremum there exists $d \in [a, c)$ such that
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- As f is increasing, $z - \epsilon < f(d) \leq f(x) \leq z$ for $d \leq x < c$.
- In other words, $0 \leq z - f(x) < \epsilon$ for $x \in (d, c)$.
- Taking $\delta = c - d$ we have $(d, c) = (c - \delta, c)$ and

Continuation

- Proof. (i) Suppose f is increasing and $c \in (a, b]$.
- Now $\{f(x) : x \in [a, c)\}$ is non-empty and is bounded above by $f(c)$. Hence it has a supremum. Take

$$z = \sup\{f(x) : x \in [a, c)\}.$$

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- Taking $\delta = c - d$ we have $(d, c) = (c - \delta, c)$ and
- $|z - f(x)| < \epsilon$ for all $x \in (c - \delta, c)$.

Continuation

- Proof. (i) Suppose f is increasing and $c \in (a, b]$.
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Continuation

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- The proofs of other claims are similar.

Continuation

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- Now $\{f(x) : x \in [a, c)\}$ is non-empty and is bounded above by $f(c)$. Hence it has a supremum. Take
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- Consider any $\epsilon > 0$. Since $z - \epsilon$ is less than the supremum there exists $d \in [a, c)$ such that
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- This proves that
$$z = \sup\{f(x) : x \in [a, c)\}.$$
- The proofs of other claims are similar.
- END OF LECTURE 27.