ANALYSIS -I

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Lecture 28. Inverses of continuous bijections and extensions of functions

- Definition 27.1: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is said to be a cluster point (or accumulation point) of A if for every $\delta > 0$

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- Proposition 27.3: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is a cluster point of A if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A\backslash\{c\}$ converging to c.

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- Note that we are excluding c from these sequences.

Limits of functions at cluster points

- Definition 27.4: Let c be a cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then f is said to have a limit at c if there exists $z \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - z| < \epsilon, \quad \forall x \in (c - \delta, c + \delta) \bigcap (A \backslash \{c\}).$$

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- Proposition 27.5: Let c be a cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then z is limit of f at c if and only if for every sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \backslash \{c\}$ converging to c, $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to z.

Left and right hand cluster points

- Definition 27.7: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is said to be a right cluster point of A if for every $\delta > 0$

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Similarly c is said to be a left cluster point of A if for every $\delta > 0$

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- (i) c is a right cluster point of A.
- (ii) There exists a sequence $\{x_n\}$ in $A \bigcap (c, \infty)$ converging to c.

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- (i) c is a right cluster point of A.
- (ii) There exists a sequence $\{x_n\}$ in $A \bigcap (c, \infty)$ converging to c.
- (iii) There exists a strictly decreasing sequence $\{x_n\}$ in A converging to c.

Monotonic functions

- Theorem 27.11: Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a function. Suppose f is increasing then the following hold.

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- (iii) For every $c \in (a, b)$

$$\lim_{x \to c-} f(x) \leq f(c) \leq \lim_{x \to c+} f(x).$$

  Therefore f is continuous at c if and only if

$$\lim_{x \to c-} f(x) = \lim_{x \to c+} f(x).$$

Inverses of monotone continuous functions

- Theorem 28.1: Let a, b, a', b' be real numbers with $a < b$ and $a' < b'$. Let $f : [a, b] \to [a', b']$ be a continuous bijection with $f(a) = a'$ and $f(b) = b'$. Then $f^{-1} : [a', b'] \to [a, b]$ is a continuous bijection.

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- Also $f^{-1}(a') = a$ and $f^{-1}(b') = b$.
- Further, we know that f is strictly increasing.
- This implies, that f^{-1} is also strictly increasing as for $y < y'$ if $f^{-1}(y) \geq f^{-1}(y')$, on applying f we get $y \geq y'$, contradicting $y < y'$.

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- Then for any $c' \in (a', b']$

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- Hence, $\lim_{y \to c'-} f^{-1}(y) = f^{-1}(c)$.
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- Hence, $\lim_{y \to c'-} f^{-1}(y) = f^{-1}(c)$.
- Similarly, for every $c' \in [a', b')$, $\lim_{y \to c+} f^{-1}(y) = f^{-1}(c')$.
- Therefore f^{-1} is continuous.

n^{th}-root function

- Example 28.2: For any $n \in \mathbb{N}$, and any $T > 0$, the function $p : [0, T] \to [0, T^n]$ defined by $p(x) = x^n$ is a continuous bijection.

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- Hence $q = p^{-1} : [0, T^n] \to [0, T]$ defined by $q(y) = y^{\frac{1}{n}}$ is a continuous bijection.
- It follows that $q : [0, \infty) \to [0, \infty)$ defined by $q(x) = x^{\frac{1}{n}}$ is a continuous bijection.

Extensions of uniformly continuous functions

- Theorem 28.3: Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : (a, b) \to \mathbb{R}$ be a function. Then there exists unique continuous function $\tilde{f} : [a, b] \to \mathbb{R}$ such that $\tilde{f}(x) = f(x), \quad \forall x \in (a, b)$ if and only if f is uniformly continuous.

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- Proof. If $\tilde{f}$ exists as above, then $\tilde{f}$ is uniformly continuous.
- This clearly implies that $f = \tilde{f}|_{(a,b)}$ is uniformly continuous.
- To prove the converse we need a lemma which is of independent interest.

Cauchy property

- Lemma 28.4: Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be uniformly continuous. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in A. Then $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

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- Now as $\{x_n\}$ is Cauchy, there exists $K \in \mathbb{N}$ such that

$$|x_m - x_n| < \delta, \quad \forall m, n \geq K.$$

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- This proves that $\{f(x_n)\}$ is Cauchy.

Continuation of proof

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- Suppose $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ are two sequences in (a, b) converging to a.

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- Since they are convergent, by the previous Lemma $\{f(x_n)\}$ and $\{f(y_n)\}$ are Cauchy.

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- Take $c = \lim_{n\to\infty} f(x_n)$ and $d = \lim_{n\to\infty} f(y_n)$.

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- Now suppose $f : (a, b) \to \mathbb{R}$ is uniformly continuous. We want to have an extension $\tilde{f} : [a, b] \to \mathbb{R}$ which is continuous.
- This means that we need to determine $\tilde{f}(a)$ and $\tilde{f}(b)$.
- Suppose $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are two sequences in (a, b) converging to a.
- Since they are convergent, by the previous Lemma $\{f(x_n)\}$ and $\{f(y_n)\}$ are Cauchy.
- Now since all Cauchy sequences in $\mathbb{R}$ are convergent these sequences are convergent.
- Take $c = \lim_{n\to\infty} f(x_n)$ and $d = \lim_{n\to\infty} f(y_n)$.
- We claim $c = d$.

Continuation

- Consider the sequence

$$z_n = \begin{cases} x_n & \text{if } n \text{ is odd;} \\ y_n & \text{if } n \text{ is even.} \end{cases}$$

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- As both $\{x_n\}$ and $\{y_n\}$ converge to the same value (namely a), $\{z_n\}$ is also convergent and it converges to a (Show this).

Continuation

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- It follows that $\{f(z_n)\}$ is also convergent.
- It has two subsequences $\{f(z_{2n-1})\}$ and $\{f(z_{2n})\}$ converging to c, d respectively. Hence $c = d = \lim_{n\to\infty} f(z_n)$.

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- We have shown that whenever a sequence $\{x_n\}$ converges to a, $\{f(x_n)\}$ is convergent and the limit is independent of the sequence chosen. Take this limit as the value of $\tilde{f}(a)$.

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- By the sequential criterion it is clear that $\tilde{f}$ defined this way is continuous at a. Similar proof works for the other cluster point b.

Continuation

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- END OF LECTURE 28.