ANALYSIS -I

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A formula for π

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- More information on Madhava series: https://en.wikipedia.org/wiki/Madhava_series
- Here is link for more on ancient Indian mathematics: https://core.ac.uk/download/pdf/326681788.pdf

Lecture 29. Differentiation

- Let $A \subseteq \mathbb{R}$. Fix $c \in A$. Assume that c is a cluster point of A. Let $f : A \to \mathbb{R}$ be a function. Then define $f_c : A \backslash \{c\} \to \mathbb{R}$ by

$$f_c(x) = \frac{f(x) - f(c)}{x - c}, \quad x \in A \backslash \{c\}.$$

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- More formally, we have the following definition.
- Definition 29.1: Let $A \subseteq \mathbb{R}$. Let $c \in A$ be a cluster point of A. Let $f : A \to \mathbb{R}$ be a function. Then f is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In such a case, $f'(c)$ is defined as this limit. If the limit does not exist f is said to be not differentiable at c.

Example

- Example 29.2 Let $f : [0,2] \to \mathbb{R}$ be the function
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- Proof: We have,

$$\begin{aligned} \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} &= \lim_{x\to 1} \frac{x^3 - 1}{x - 1} \\ &= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} \\ &= \lim_{x\to 1} (x^2 + x + 1) \\ &= 3. \end{aligned}$$

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- Remark: We may also write $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ as
$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

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- Hence

$$\lim_{x \to c}(f(x) - f(c)) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.(x - c)$$

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- Hence f is continuous at c.
- The function $g(x) = |x|, x \in \mathbb{R}$ is continuous at 0, but is not differentiable at 0 (Why?). ■

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- (i) For $a, b \in \mathbb{R}$, $af + bg$ defined by $(af + bg)(x) = af(x) + bg(x), \quad x \in I$ is differentiable at c and,

$$(af + bg)'(c) = af'(c) + bg'(c).$$

- (ii) The product fg defined by $fg(x) = f(x)g(x), \quad x \in I$, is differentiable at c and

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- Proof. (i) The proof is clear.

Continuation

- (ii) We have

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- Now taking limit as x tends to c in the previous equation, we see that (fg) is differentiable at c and

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- (iii) As g is continuous at c and $g(c) \neq 0$, $g(x) \neq 0$ for some interval J containing c. Hence $\frac{f}{g}$ is defined in this interval.

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- Now

$$\frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} = \frac{1}{g(x)g(c)} \frac{f(x)g(c) - f(c)g(x)}{x - c}$$
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- That completes the proof.

Polynomials

- Theorem 29.5: Let $p : \mathbb{R} \to \mathbb{R}$ be a real polynomial:

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, x \in \mathbb{R}$$

for some $n \in \mathbb{N}, a_0, a_1, \ldots, a_n \in \mathbb{R}$.

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- Then at any $c \in \mathbb{R}$ p is differentiable at c and

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- Then at any $c \in \mathbb{R}$ p is differentiable at c and
$$p'(c) = a_1 + 2a_2 c + 3a_3 c^2 + \cdots + na_n c^{(n-1)}.$$
- Proof. This can be proved using (i) and (ii) of previous theorem and induction. More directly:
$$\begin{aligned} & p'(c) \\ = \ & \lim_{h\to 0} \frac{p(h+c) - p(h)}{h} \\ = \ & \lim_{h\to 0} \frac{1}{h}[a_1.h + a_2((h+c)^2 - c^2)) + a_3(h+c)^3 - c^3) \\ & \quad + \cdots + a_n((h+c)^n - c^n) \\ = \ & a_1 + 2a_2 c + 3a_3 c^2 + \cdots + na_n c^{(n-1)}. \end{aligned}$$

Differentiable functions

- Definition 29.6: A function $f : I \to \mathbb{R}$ is said to be differentiable if it is differentiable at every $c \in I$. If $f : I \to \mathbb{R}$ is differentiable then the function $f' : I \to \mathbb{R}$ is called the first derivative of f.

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- END OF LECTURE 29.