ANALYSIS -I

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 3: Sets and functions

- Informal Definition: A set is a collection of well-defined objects.

Lecture 3: Sets and functions

- Informal Definition: A set is a collection of well-defined objects.
- We continue with this definition though ideally speaking we should be following ZFC axioms.

Lecture 3: Sets and functions

- Informal Definition: A set is a collection of well-defined objects.
- We continue with this definition though ideally speaking we should be following ZFC axioms.
- We assume familiarity with

Lecture 3: Sets and functions

- Informal Definition: A set is a collection of well-defined objects.
- We continue with this definition though ideally speaking we should be following ZFC axioms.
- We assume familiarity with
- $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers.

Lecture 3: Sets and functions

- Informal Definition: A set is a collection of well-defined objects.
- We continue with this definition though ideally speaking we should be following ZFC axioms.
- We assume familiarity with
- $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$-the set of integers.

Functions

- Given two non-empty sets A and B, a function f from A to B is an association of some element $f(x)$ in B, for every x in A.

Functions

- Given two non-empty sets A and B, a function f from A to B is an association of some element $f(x)$ in B, for every x in A.
- This is denoted by $f : A \to B$.

Functions

- Given two non-empty sets A and B, a function f from A to B is an association of some element $f(x)$ in B, for every x in A.
- This is denoted by $f : A \to B$.
- You may also think of a function f as a subset of the Cartesian product $A \times B = \{(a, b) : a \in A, b \in B\}$ having certain properties.

Functions

- Given two non-empty sets A and B, a function f from A to B is an association of some element $f(x)$ in B, for every x in A.
- This is denoted by $f : A \to B$.
- You may also think of a function f as a subset of the Cartesian product $A \times B = \{(a, b) : a \in A, b \in B\}$ having certain properties.
- More precisely, $G(f) = \{(x, f(x)) : x \in A\}$ is a subset of $A \times B$, where every element $x \in A$ appears with exactly one element $f(x) \in B$.

Functions

- Given two non-empty sets A and B, a function f from A to B is an association of some element $f(x)$ in B, for every x in A.
- This is denoted by $f : A \to B$.
- You may also think of a function f as a subset of the Cartesian product $A \times B = \{(a, b) : a \in A, b \in B\}$ having certain properties.
- More precisely, $G(f) = \{(x, f(x)) : x \in A\}$ is a subset of $A \times B$, where every element $x \in A$ appears with exactly one element $f(x) \in B$.
- $G(f)$ is known as the graph of f.

Vertical line test

- Clearly not all subsets G of $A \times B$ appear as graphs of f.

Vertical line test

- Clearly not all subsets G of $A \times B$ appear as graphs of f.
- Every element $x \in A$ should appear. More over for every element x there should be unique x' in B such that $(x, x') \in G$.

Vertical line test

- Clearly not all subsets G of $A \times B$ appear as graphs of f.
- Every element $x \in A$ should appear. More over for every element x there should be unique x' in B such that $(x, x') \in G$.
- In other words, there should not be x', x'' in B with $x' \neq x''$, such that both $(x.x')$ and (x, x'') are in G.

Vertical line test

- Clearly not all subsets G of $A \times B$ appear as graphs of f.
- Every element $x \in A$ should appear. More over for every element x there should be unique x' in B such that $(x, x') \in G$.
- In other words, there should not be x', x'' in B with $x' \neq x''$, such that both $(x.x')$ and (x, x'') are in G.
- In the usual picture of graphs of functions on real line this is known as vertical line test. A graph of a function can not be touching a vertical line at more than one point.

function as a machine

- You may think of a function $f : A \to B$ as a machine.

function as a machine

- You may think of a function $f : A \to B$ as a machine.
- It takes any $x \in A$ as input and spews out some element $f(x)$ in B as out put.

function as a machine

- You may think of a function $f : A \to B$ as a machine.
- It takes any $x \in A$ as input and spews out some element $f(x)$ in B as out put.
- Any element of A can be input.

function as a machine

- You may think of a function $f : A \to B$ as a machine.
- It takes any $x \in A$ as input and spews out some element $f(x)$ in B as out put.
- Any element of A can be input.
- With one input there is only one output.

function as a machine

- You may think of a function $f : A \to B$ as a machine.
- It takes any $x \in A$ as input and spews out some element $f(x)$ in B as out put.
- Any element of A can be input.
- With one input there is only one output.
- Different inputs may give same output.

function as a machine

- You may think of a function $f : A \to B$ as a machine.
- It takes any $x \in A$ as input and spews out some element $f(x)$ in B as out put.
- Any element of A can be input.
- With one input there is only one output.
- Different inputs may give same output.
- Some elements of B may not be an output value for f.

Domain, Co-domain and Range

- Terminology: Suppose $f : A \rightarrow B$ is a function.

Domain, Co-domain and Range

- Terminology: Suppose $f : A \to B$ is a function.
- Then A is known as the domain of f.

Domain, Co-domain and Range

- Terminology: Suppose $f : A \to B$ is a function.
- Then A is known as the domain of f.
- B is known as the co-domain of f.

Domain, Co-domain and Range

- Terminology: Suppose $f : A \to B$ is a function.
- Then A is known as the domain of f.
- B is known as the co-domain of f.
- The set $\{f(x) : x \in A\}$ is known as the range of f.

Domain, Co-domain and Range

- Terminology: Suppose $f : A \to B$ is a function.
- Then A is known as the domain of f.
- B is known as the co-domain of f.
- The set $\{f(x) : x \in A\}$ is known as the range of f.
- Note that the range of f is a subset of the co-domain.

Domain, Co-domain and Range

- Terminology: Suppose $f : A \to B$ is a function.
- Then A is known as the domain of f.
- B is known as the co-domain of f.
- The set $\{f(x) : x \in A\}$ is known as the range of f.
- Note that the range of f is a subset of the co-domain.
- Sometimes people call B, the co-domain as range of f. It is better to avoid that kind of terminology as it can lead to confusion.

Students and Hostel rooms

- Think of A as students and B as the set of hostel rooms.

Students and Hostel rooms

- Think of A as students and B as the set of hostel rooms.
- Then think of a function $f : A \to B$ as allotment of rooms. In other words, student x gets room $f(x)$.

Students and Hostel rooms

- Think of A as students and B as the set of hostel rooms.
- Then think of a function $f : A \to B$ as allotment of rooms. In other words, student x gets room $f(x)$.
- Note that to have a genuine function f it is necessary that all students are allotted rooms. Nobody is left out.

Students and Hostel rooms

- Think of A as students and B as the set of hostel rooms.
- Then think of a function $f : A \to B$ as allotment of rooms. In other words, student x gets room $f(x)$.
- Note that to have a genuine function f it is necessary that all students are allotted rooms. Nobody is left out.
- Same student can't be allotted multiple rooms. In other words if $y = f(x)$ and $z = f(x)$, then $y = z$.

Students and Hostel rooms

- Think of A as students and B as the set of hostel rooms.
- Then think of a function $f : A \to B$ as allotment of rooms. In other words, student x gets room $f(x)$.
- Note that to have a genuine function f it is necessary that all students are allotted rooms. Nobody is left out.
- Same student can't be allotted multiple rooms. In other words if $y = f(x)$ and $z = f(x)$, then $y = z$.
- It is fine, if some rooms are vacant. In other words, there could be $y \in B$ such that $y \neq f(x)$ for any $x \in A$.

Students and Hostel rooms

- Think of A as students and B as the set of hostel rooms.
- Then think of a function $f : A \to B$ as allotment of rooms. In other words, student x gets room $f(x)$.
- Note that to have a genuine function f it is necessary that all students are allotted rooms. Nobody is left out.
- Same student can't be allotted multiple rooms. In other words if $y = f(x)$ and $z = f(x)$, then $y = z$.
- It is fine, if some rooms are vacant. In other words, there could be $y \in B$ such that $y \neq f(x)$ for any $x \in A$.
- It is also fine if students are asked to share rooms. In other words it is possible to have x, x' in A, such that $f(x) = f(x')$.

Injectivity

- Let A, B be non-empty sets and let $f : A \to B$ be a function.

Injectivity

- Let A, B be non-empty sets and let $f : A \to B$ be a function.
- Definition: Then f is said to be injective or one to one if a_1, a_2 are in A and $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$. In other words, distinct elements are mapped to distinct elements.

Injectivity

- Let A, B be non-empty sets and let $f : A \to B$ be a function.
- Definition: Then f is said to be injective or one to one if a_1, a_2 are in A and $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$. In other words, distinct elements are mapped to distinct elements.
- Equivalently, f is injective if $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

Injectivity

- Let A, B be non-empty sets and let $f : A \to B$ be a function.
- Definition: Then f is said to be injective or one to one if a_1, a_2 are in A and $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$. In other words, distinct elements are mapped to distinct elements.
- Equivalently, f is injective if $f(a_1) = f(a_2)$ implies $a_1 = a_2$.
- In the language of machines this corresponds to outputs being different for different inputs.

Injectivity

- Let A, B be non-empty sets and let $f : A \to B$ be a function.
- Definition: Then f is said to be injective or one to one if a_1, a_2 are in A and $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$. In other words, distinct elements are mapped to distinct elements.
- Equivalently, f is injective if $f(a_1) = f(a_2)$ implies $a_1 = a_2$.
- In the language of machines this corresponds to outputs being different for different inputs.
- While allotting rooms to students, injectivity or one-to-one means there is no sharing of rooms.

Surjectivity

- Let A, B be non-empty sets and let $f : A \to B$ be a function.

Surjectivity

- Let A, B be non-empty sets and let $f : A \rightarrow B$ be a function.
- Definition: Then f is said to be surjective or onto if the range of f is same as the co-domain.

Surjectivity

- Let A, B be non-empty sets and let $f : A \to B$ be a function.
- Definition: Then f is said to be surjective or onto if the range of f is same as the co-domain.
- Equivalently, f is surjective if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

Surjectivity

- Let A, B be non-empty sets and let $f : A \to B$ be a function.
- Definition: Then f is said to be surjective or onto if the range of f is same as the co-domain.
- Equivalently, f is surjective if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.
- Thinking of machines, f is surjective if every element of B can be produced using f.

Surjectivity

- Let A, B be non-empty sets and let $f : A \to B$ be a function.
- Definition: Then f is said to be surjective or onto if the range of f is same as the co-domain.
- Equivalently, f is surjective if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.
- Thinking of machines, f is surjective if every element of B can be produced using f.
- In the problem of allotting rooms to students it means that the hostel is full. That is all the rooms have got allotted.

Bijections

- Give examples to show that an injective function need not be surjective and a surjective function need not be injective.

Bijections

- Give examples to show that an injective function need not be surjective and a surjective function need not be injective.
- Definition: Let A, B be non-empty sets and let $f : A \to B$ be a function. Then f is said to be bijective if f is both injective and surjective. In other words, it is both one to one and onto.

Bijections

- Give examples to show that an injective function need not be surjective and a surjective function need not be injective.
- Definition: Let A, B be non-empty sets and let $f : A \to B$ be a function. Then f is said to be bijective if f is both injective and surjective. In other words, it is both one to one and onto.
- Define $f_1 : \mathbb{Z} \to \mathbb{Z}$ by $f_1(n) = n + 1, \quad \forall n \in \mathbb{Z}$. Then f_1 is a bijection.

Bijections

- Give examples to show that an injective function need not be surjective and a surjective function need not be injective.
- Definition: Let A, B be non-empty sets and let $f : A \to B$ be a function. Then f is said to be bijective if f is both injective and surjective. In other words, it is both one to one and onto.
- Define $f_1 : \mathbb{Z} \to \mathbb{Z}$ by $f_1(n) = n + 1, \quad \forall n \in \mathbb{Z}$. Then f_1 is a bijection.
- Define $f_2 : \mathbb{Z} \to \mathbb{Z}$ by $f_2(n) = -n, \quad \forall n \in \mathbb{Z}$. Then f_2 is a bijection.
- Define $f_3 : \mathbb{Z} \to \mathbb{Z}$ by $f_3(n) = n^2$. Then f_3 is neither injective nor surjective.

Compositions of functions

- Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$ be functions. Then a new function $g \circ f : A \to C$ is got by taking
$$g \circ f(a) = g(f(a)), \quad \forall a \in A.$$

Compositions of functions

- Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$ be functions. Then a new function $g \circ f : A \to C$ is got by taking
$$g \circ f(a) = g(f(a)), \quad \forall a \in A.$$
- $g \circ f$ is known as composition of g and f.

Compositions of functions

- Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$ be functions. Then a new function $g \circ f : A \to C$ is got by taking
$$g \circ f(a) = g(f(a)), \quad \forall a \in A.$$
- $g \circ f$ is known as composition of g and f.
- The out put of machine f is taken as input for g.

Inverse map

- Let A, B be non-empty sets and let $f : A \to B$ be a bijection. Then we see that for every $b \in B$ there exists unique $a \in A$ such that $f(a) = b$. Then we call a as $f^{-1}(b)$.

Inverse map

- Let A, B be non-empty sets and let $f : A \to B$ be a bijection. Then we see that for every $b \in B$ there exists unique $a \in A$ such that $f(a) = b$. Then we call a as $f^{-1}(b)$.
- In other words, if $f : A \to B$ is a bijection then there exists a unique function $f^{-1} : B \to A$ such that

$$f \circ f^{-1}(b) = b, \quad \forall b \in B$$

and

$$f^{-1} \circ f(a) = a, \quad \forall a \in A.$$

Inverse map

- Let A, B be non-empty sets and let $f : A \to B$ be a bijection. Then we see that for every $b \in B$ there exists unique $a \in A$ such that $f(a) = b$. Then we call a as $f^{-1}(b)$.
- In other words, if $f : A \to B$ is a bijection then there exists a unique function $f^{-1} : B \to A$ such that

$$f \circ f^{-1}(b) = b, \quad \forall b \in B$$

and

$$f^{-1} \circ f(a) = a, \quad \forall a \in A.$$

- So $f \circ f^{-1}$ is the identity map on B and $f^{-1} \circ f$ is the identity map on A.

Inverse map

- Let A, B be non-empty sets and let $f : A \to B$ be a bijection. Then we see that for every $b \in B$ there exists unique $a \in A$ such that $f(a) = b$. Then we call a as $f^{-1}(b)$.
- In other words, if $f : A \to B$ is a bijection then there exists a unique function $f^{-1} : B \to A$ such that

$$f \circ f^{-1}(b) = b, \quad \forall b \in B$$

and

$$f^{-1} \circ f(a) = a, \quad \forall a \in A.$$

- So $f \circ f^{-1}$ is the identity map on B and $f^{-1} \circ f$ is the identity map on A.
- The identity map is a completely lazy machine where the output is same as the input.

One sided inverses

- Example: Suppose $A = \{x, y\}$ and $B = \{4, 5, 6\}$.

One sided inverses

- Example: Suppose $A = \{x, y\}$ and $B = \{4, 5, 6\}$.
- Define $f : A \to B$ by $f(x) = 4$ and $f(y) = 6$.

One sided inverses

- Example: Suppose $A = \{x, y\}$ and $B = \{4, 5, 6\}$.
- Define $f : A \to B$ by $f(x) = 4$ and $f(y) = 6$.
- Define $g : B \to A$ by $g(4) = g(5) = x$ and $g(6) = y$.

One sided inverses

- Example: Suppose $A = \{x, y\}$ and $B = \{4, 5, 6\}$.
- Define $f : A \to B$ by $f(x) = 4$ and $f(y) = 6$.
- Define $g : B \to A$ by $g(4) = g(5) = x$ and $g(6) = y$.
- Then $g \circ f(x) = x$ and $g \circ f(y) = y$.

One sided inverses

- Example: Suppose $A = \{x, y\}$ and $B = \{4, 5, 6\}$.
- Define $f : A \to B$ by $f(x) = 4$ and $f(y) = 6$.
- Define $g : B \to A$ by $g(4) = g(5) = x$ and $g(6) = y$.
- Then $g \circ f(x) = x$ and $g \circ f(y) = y$.
- So $g \circ f$ is the identity map on A. However, $f \circ g$ is not the identity map on B.

Properties inferred from compositions

- Let A, B, C be non-empty sets and let $f : A \to B$ and $g : B \to C$ be functions.

Properties inferred from compositions

- Let A, B, C be non-empty sets and let $f : A \to B$ and $g : B \to C$ be functions.
- Theorem 3.1: Suppose $g \circ f$ is one to one then f is one to one.

Properties inferred from compositions

- Let A, B, C be non-empty sets and let $f : A \to B$ and $g : B \to C$ be functions.
- Theorem 3.1: Suppose $g \circ f$ is one to one then f is one to one.
- Proof: Take $h = g \circ f$. Suppose $f(a_1) = f(a_2)$ for some a_1, a_2 in A. Then by the definition of a function, $g(f(a_1)) = g(f(a_2))$. In other words, $h(a_1) = h(a_2)$. But h is assumed to be one to one. Hence $a_1 = a_2$. This shows that f is one to one.

Properties inferred from compositions

- Let A, B, C be non-empty sets and let $f : A \to B$ and $g : B \to C$ be functions.
- Theorem 3.1: Suppose $g \circ f$ is one to one then f is one to one.
- Proof: Take $h = g \circ f$. Suppose $f(a_1) = f(a_2)$ for some a_1, a_2 in A. Then by the definition of a function, $g(f(a_1)) = g(f(a_2))$. In other words, $h(a_1) = h(a_2)$. But h is assumed to be one to one. Hence $a_1 = a_2$. This shows that f is one to one.
- Theorem 3.2: Suppose $g \circ f$ is onto then g is onto.

Properties inferred from compositions

- Let A, B, C be non-empty sets and let $f : A \to B$ and $g : B \to C$ be functions.
- Theorem 3.1: Suppose $g \circ f$ is one to one then f is one to one.
- Proof: Take $h = g \circ f$. Suppose $f(a_1) = f(a_2)$ for some a_1, a_2 in A. Then by the definition of a function, $g(f(a_1)) = g(f(a_2))$. In other words, $h(a_1) = h(a_2)$. But h is assumed to be one to one. Hence $a_1 = a_2$. This shows that f is one to one.
- Theorem 3.2: Suppose $g \circ f$ is onto then g is onto.
- Proof: Exercise!

Powers of functions

- Let A be a non-empty set and let $f : A \to A$ be a function.

Powers of functions

- Let A be a non-empty set and let $f : A \to A$ be a function.
- Then $f^2 : A \to A$ is defined as $f^2(a) = f \circ f(a) = f(f(a))$.

Powers of functions

- Let A be a non-empty set and let $f : A \to A$ be a function.
- Then $f^2 : A \to A$ is defined as $f^2(a) = f \circ f(a) = f(f(a))$.
- Similarly $f^3(a) = (f \circ f \circ f)(a) = f(f(f(a)))$.

Powers of functions

- Let A be a non-empty set and let $f : A \to A$ be a function.
- Then $f^2 : A \to A$ is defined as $f^2(a) = f \circ f(a) = f(f(a))$.
- Similarly $f^3(a) = (f \circ f \circ f)(a) = f(f(f(a)))$.
- More generally, we can define f^n for any natural number n.

Powers of functions

- Let A be a non-empty set and let $f : A \to A$ be a function.
- Then $f^2 : A \to A$ is defined as $f^2(a) = f \circ f(a) = f(f(a))$.
- Similarly $f^3(a) = (f \circ f \circ f)(a) = f(f(f(a)))$.
- More generally, we can define f^n for any natural number n.
- Note that in general you can not define f^2 when f is a function from one set to a different set.

Conway's problem

- Consider $h : \mathbb{Z} \to \mathbb{Z}$ defined by

$$h(n) = \begin{cases} 3k & \text{if} \quad n = 2k, \quad k \in \mathbb{Z} \\ 3k+1 & \text{if} \quad n = 4k+1 \quad k \in \mathbb{Z} \\ 3k-1 & \text{if} \quad n = 4k-1 \quad k \in \mathbb{Z} \end{cases}$$

Conway's problem

- Consider $h : \mathbb{Z} \to \mathbb{Z}$ defined by

$$h(n) = \begin{cases} 3k & \text{if} \quad n = 2k, \quad k \in \mathbb{Z} \\ 3k+1 & \text{if} \quad n = 4k+1 \quad k \in \mathbb{Z} \\ 3k-1 & \text{if} \quad n = 4k-1 \quad k \in \mathbb{Z} \end{cases}$$

- Here on the repeated action of h,

$$7 \to 5 \to 4 \to 6 \to 9 \to 7.$$

Conway's problem

- Consider $h : \mathbb{Z} \to \mathbb{Z}$ defined by

$$h(n) = \begin{cases} 3k & \text{if} \quad n = 2k, \quad k \in \mathbb{Z} \\ 3k+1 & \text{if} \quad n = 4k+1 \quad k \in \mathbb{Z} \\ 3k-1 & \text{if} \quad n = 4k-1 \quad k \in \mathbb{Z} \end{cases}$$

- Here on the repeated action of h,

$$7 \to 5 \to 4 \to 6 \to 9 \to 7.$$

- So we end up with a loop or a 'cycle'.

Conway's problem

- Consider $h : \mathbb{Z} \to \mathbb{Z}$ defined by

$$h(n) = \begin{cases} 3k & \text{if} \quad n = 2k, \quad k \in \mathbb{Z} \\ 3k+1 & \text{if} \quad n = 4k+1 \quad k \in \mathbb{Z} \\ 3k-1 & \text{if} \quad n = 4k-1 \quad k \in \mathbb{Z} \end{cases}$$

- Here on the repeated action of h,

$$7 \to 5 \to 4 \to 6 \to 9 \to 7.$$

- So we end up with a loop or a 'cycle'.
- Show that h is a bijection.
- Challenge Problem 2: What happens if we start with 8? Do we ever come back to 8, that is, is there a cycle starting at 8?

Conway's problem

- Consider $h : \mathbb{Z} \to \mathbb{Z}$ defined by

$$h(n) = \begin{cases} 3k & \text{if} \quad n = 2k, \quad k \in \mathbb{Z} \\ 3k+1 & \text{if} \quad n = 4k+1 \quad k \in \mathbb{Z} \\ 3k-1 & \text{if} \quad n = 4k-1 \quad k \in \mathbb{Z} \end{cases}$$

- Here on the repeated action of h,

$$7 \to 5 \to 4 \to 6 \to 9 \to 7.$$

- So we end up with a loop or a 'cycle'.
- Show that h is a bijection.
- Challenge Problem 2: What happens if we start with 8? Do we ever come back to 8, that is, is there a cycle starting at 8?
- END OF LECTURE 3.