ANALYSIS -I

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- Definition 29.1: Let $A \subseteq \mathbb{R}$. Let $c \in A$ be a cluster point of A. Let $f : A \to \mathbb{R}$ be a function. Then f is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In such a case, $f'(c)$ is defined as this limit. If the limit does not exist f is said to be not differentiable at c.

Chain rule

- Theorem 30.1 Let I, J be intervals and let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be functions such that $f(I) \subseteq J$ and $h = g \circ f$. Consider $c \in I$. Suppose f is differentiable at c and g is differentiable at $f(c)$. Then h is differentiable at c and

$$h'(c) = (g \circ f)'(c) = g'(f(c))f'(c).$$

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- Rough computation:

$$\frac{g \circ f(x) - g \circ f(c)}{x - c} = \frac{g \circ f(x) - g \circ f(c)}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}$$

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- Taking limit as x tends to c we should get the answer as $f(x)$ converges to $f(c)$.
- However, there is a problem here as we can't ensure that $f(x) - f(c) \neq 0$.

Caratheodory's theorem

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- Theorem 30.2: Let $f : I \to \mathbb{R}$ be a function where I is an interval. Fix $c \in I$. Then f is differentiable at c if and only if there exists a function $u : I \to \mathbb{R}$ such that

$$f(x) - f(c) = (x - c)u(x), \quad \forall x \in I \qquad (*)$$

and u is continuous at c. In such a case, $u(c) = f'(c)$.

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- Proof: If f is differentiable at c, take
$$u(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & \text{if } x \neq c, x \in I \\ f'(c) & \text{if } x = c. \end{cases}$$

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- Then it is easy to see that $(*)$ is satisfied and u is continuous at c.
- Conversely if u exists satisfying $(*)$ and u is continuous at c
- From $(*)$, $u(x) = \frac{f(x)-f(c)}{x-c}$ for $x \neq c$. Taking limit as x tends to c, using continuity of u at c, f is differentiable at c, and $u(c) = f'(c)$. ■

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- Since $f(I) \subseteq J$, this equation is also true at $y = f(x)$ and so we get

$$g(f(x)) - g(f(c)) = (f(x) - f(c))v(f(x)), \quad \forall x \in I.$$

Continuation

- Now using the previous equation, we have

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- Hence by Caratheodory's theorem, $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = u(c)v(f(c)) = f'(c)g'(f(c)).$$

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$$(g \circ f)'(c) = u(c)v(f(c)) = f'(c)g'(f(c)).$$

- In other words $h'(c) = g'(f(c))f'(c)$. $\blacksquare$.

Derivative of inverse -I

- Theorem 30.3: Let I, J be intervals and let $f : I \to J$ be a bijection. Suppose f is differentiable at $c \in I$ and $g := f^{-1}$ is differentiable at $f(c)$. Then

$$g'(f(c)) = \frac{1}{f'(c)}.$$

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- Now by the chain rule we get $1 = h'(c) = f'(c)g'(f(c))$.
- Consequently, $g'(f(c)) = \frac{1}{f'(c)}$.$\blacksquare$
- Note that this in particular means that in this Theorem, $f'(c) = 0$ is not possible.

Derivative of inverse -II

- Theorem 30.4: Let I, J be intervals and let $f : I \to J$ be a bijection. Suppose f is differentiable at $c \in I$ and $f'(c) \neq 0$. Also assume that f^{-1} is continuous at $f(c)$. Then $g := f^{-1}$ is differentiable at $f(c)$ and $g'(f(c)) = \frac{1}{f'(c)}$.

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- Proof: By Caratheodory's theorem, there exists a function u on I, which is continuous at c and

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- First we note that $u(x) \neq 0$ for every x. Indeed, for $x \neq c$, $f(x) \neq f(c)$ as f is injective and hence $u(x) \neq 0$. At $x = c$, $u(c) = f'(c)$, which is not zero by hypothesis.

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- Now take $y = f(x)$ and $d = f(c)$ in the equation above, to get

$$y - d = (f^{-1}(x) - f^{-1}(d))u(f^{-1}(y))$$

Continuation

- Since f is surjective, this equation is true for every $y \in J$ and we get
$$g(y) - g(d) = (y - d)\left(\frac{1}{u(g(y))}\right).$$

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- Example 30.5: For $n \in \mathbb{N}$ the function $g : (0, \infty) \to (0, \infty)$ defined by $g(y) = y^{\frac{1}{n}}$ is differentiable and
$$g'(y) = \frac{1}{ny^{1-\frac{1}{n}}}, \quad y \in (0, \infty).$$

Local extremums

- Definition 30.6: Let $f : I \to \mathbb{R}$ be a function and suppose $c \in I$. Then c is said to be a local maximum of f if there exists $\delta > 0$ such that

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- If c is a local maximum or local minimum it is said to be a local extremum.

Global extremums

- Definition 30.7: Let $f : I \to \mathbb{R}$ be a function and suppose $c \in I$. Then c is said to be a global maximum of f if

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Interior extremum theorem

- Definition 30.8: Let I be an interval and let $c \in I$. Then c is said to be an interior point of I if there exists $\delta > 0$ such that

$$(c - \delta, c + \delta) \subseteq I.$$

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- Theorem 30.9: Let $f : I \to \mathbb{R}$ be a function. Suppose c is an interior point of I and suppose c is a local extremum of f. If f is differentiable at c then

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- So there exists $\delta_1 > 0$ such that $(c-\delta_1, c+\delta_1) \subseteq I$.

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- So there exists $\delta_1 > 0$ such that $(c-\delta_1, c+\delta_1) \subseteq I$.
- Suppose that c is a local maximum of f. Then there exists $\delta_2 > 0$ such that
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$$f(c) \geq f(x) \quad \forall x \in (c-\delta_2, c+\delta_2) \bigcap I.$$
- Taking $\delta = \min\{\delta_1, \delta_2\}$, we have $(c-\delta, c+\delta) \subseteq I$ and
$$f(c) \geq f(x), \quad \forall x \in (c-\delta, c+\delta).$$

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- Then for every n, $x_n > c$ and $f(x_n) \leq f(c)$ and hence

$$\frac{f(x_n) - f(c)}{x_n - c} \leq 0 \qquad (1)$$

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- Taking limit as $n \to \infty$, we get

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- Now suppose $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in $(c-\delta, c)$ converging to c (For instance, we can take $y_n = c - \frac{\delta}{2n}$.)

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$$f'(c) \geq 0. \qquad (2)$$

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$$\frac{f(y_n) - f(c)}{y_n - c} \geq 0$$

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$$f'(c) \geq 0. \qquad (2)$$

- Combining inequalities (1) and (2) we get $f'(c) = 0$ as required. ■

Rolle's theorem

- Theorem 30.10 (Rolle's theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Suppose $f(a) = f(b) = 0$. Then there exists $c \in (a, b)$ such that

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- Suppose there exists some $t \in (a, b)$ such that $f(t) > 0$, then as $f(a) = f(b) = 0$, the global maximum of f is attained at some $c \in (a, b)$.

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- Proof: Since f is continuous on $[a, b]$, f attains global maximum and global minimum in $[a, b]$.
- Suppose there exists some $t \in (a, b)$ such that $f(t) > 0$, then as $f(a) = f(b) = 0$, the global maximum of f is attained at some $c \in (a, b)$.
- In particular, c is a local extremum and by the interior extremum theorem, $f'(c) = 0$ and we are done.

Rolle's theorem

- Theorem 30.10 (Rolle's theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Suppose $f(a) = f(b) = 0$. Then there exists $c \in (a, b)$ such that
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- Similarly, if there exists $s \in (a, b)$ such that $f(s) < 0$ then global minimum is attained in (a, b) and if d is one such point, then $f'(d) = 0$.
- The only other possibility is $f(x) = 0$ for all $x \in [a, b]$ and in such a case $f'(x) = 0$ for all $x \in (a, b)$ and we are done. $\blacksquare$.

Example

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- END OF LECTURE 30