ANALYSIS -I

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Lecture 31. Mean value theorem

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Lecture 31. Mean value theorem

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- Definition 29.1: Let $A \subseteq \mathbb{R}$. Let $c \in A$ be a cluster point of A. Let $f : A \to \mathbb{R}$ be a function. Then f is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In such a case, $f'(c)$ is defined as this limit. If the limit does not exist f is said to be not differentiable at c.

Interior Extremum theorem and Rolle's theorem

- Theorem 30.9 (Interior Extremum theorem): Let $f : I \to \mathbb{R}$ be a function. Suppose c is an interior point of I and suppose c is a local extremum of f. If f is differentiable at c then

$$f'(c) = 0.$$

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- Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence decreasing to c. Then

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- Similarly if $\{y_n\}_{n\in\mathbb{N}}$ is a sequence increasing to c,

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- Combining two inequalities we get $f'(c) = 0$.

Rolle's theorem

- Theorem 30.10 (Rolle's theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Suppose $f(a) = f(b) = 0$. Then there exists $c \in (a, b)$ such that

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- If f is non-zero it attains either supremum or infimum at some interior point c in (a, b).
- Then by interior extremum theorem $f'(c) = 0$.

Mean value theorem (MVT)

- Theorem 31.1 (Mean value theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Then there exists $c \in (a, b)$ such that

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$$g(a) = g(b) = 0.$$

- Hence Rolle's theorem is applicable to g, and we get $c \in (a, b)$ such that $g'(c) = 0$.

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- Note that Rolle's theorem is a special case of mean value theorem.

Cauchy's mean value theorem

- Theorem 31.2 (Cauchy's Mean value theorem): Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions which are differentiable on (a, b). Then there exists $c \in (a, b)$ such that
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- Define $h : [a, b] \to \mathbb{R}$ by
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- Therefore Rolle's theorem is applicable.

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- So we get $c \in (a, b)$ such that $h'(c) = 0$ and that gives the result.

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- So we get $c \in (a, b)$ such that $h'(c) = 0$ and that gives the result.
- Note that mean value theorem is a special case of Cauchy's mean value theorem with $g(x) = x, \quad x \in [a, b]$.

Applications of mean value theorem

- Corollary 31.3: Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on (a, b). Suppose $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant.

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- Clearly mean value theorem is applicable to this function and we get

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$$f(t) - f(a) = 0.(t - a) = 0.$$
- Therefore $f(t) = f(a)$.
- In other words $f(t) = f(a)$ for every $t \in [a, b]$.■

Equal derivatives

- Corollary 31.4: Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions differentiable on (a, b). Suppose $f'(x) = g'(x)$ for all $x \in (a, b)$. Then $f(x) = g(x) + C, \quad x \in [a, b]$ for some $C \in \mathbb{R}$.

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- Proof: This is clear from the previous corollary, by considering the function, $h : [a, b] \to \mathbb{R}$ defined by

$$h(x) = f(x) - g(x), \quad x \in [a, b].$$

Monotonicity

- Recall that a function $f : [a, b] \to \mathbb{R}$ is said to be increasing (respectively decreasing) if $f(x) \leq f(y)$ (respectively $f(x) \geq f(y)$) for all x, y in $[a, b]$ with $x \leq y$.

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- Theorem 31.5: Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b).
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- Proof: (i) Suppose f is increasing and $x \in (a, b)$.
- Consider any sequence $\{x_n\}$ in (a, b) with $x < x_n \leq b$, converging to x. Then $f(x_n) - f(x) \geq 0$ for all n and we get

$$f'(x) = \lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} \geq 0.$$

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- So we get
$$f(y) - f(x) = f'(z)(y - x)$$
- for some $z \in [x, y]$. Then by the hypothesis, $f'(z) \geq 0$ and therefore $f(y) - f(x) \geq 0$ or $f(y) \geq f(x)$.

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- Proof of (ii) is similar. ■

Strictly increasing functions

- Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose $f'(x) > 0$ for all $x \in (a, b)$ then by mean value theorem it is easy to see that f is strictly increasing.

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- Example 31.6: Consider $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = x^3, \quad x \in [-1, 1].$$

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- Remark 31.7: In this Example, 0 is a point which is never picked up by the mean value theorem. That is, for no $x, y \in [-1, 1]$ with $x < y$, $f(y) - f(x) = f'(0)(y - x)$. Can we characterize such points?

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- END OF LECTURE 31.