ANALYSIS -I

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- Let us look at a few basic properties of the set of natural numbers and its subsets.

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- Note that clearly the minimal element of R is unique, for if both k, l are minimal then we have $k \leq l$ and $l \leq k$, and this means $k = l$.
- We also note that if $n \in R$, then the minimal element of R is contained in $\{1, 2, \ldots, n\} \bigcap R$. So the existence of minimum here is essentially a statement about finite sets.

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- Proof: (1) $\Rightarrow$ (2). Assume well ordering principle. Now suppose S is a subset of $\mathbb{N}$ satisfying (i) and (ii). We want to show that $S = \mathbb{N}$. Suppose not. Then $R = \mathbb{N} \backslash S$ is non-empty.

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- By well ordering principle, R has a minimal element, say $m \in R$.
- Now $m \neq 1$ as $1 \in S$. Therefore, $m - 1 \in \mathbb{N}$. As m is the minimal element of R, $m - 1 \in S$. By property (ii), this yields, $m = (m - 1) + 1 \in S$. This is a contradiction as $m \in R$ and $R \bigcap S = \emptyset$.

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- Hence $S = \mathbb{N}$.

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- We want to show that $T = \mathbb{N}$.
- Take $S = \{m \in \mathbb{N} : \{1, 2, \ldots, m\} \subseteq T\}$.
- In view of (a), $1 \in T$ and hence $1 \in S$.
- In view of (b), if $m \in S$ then $m + 1 \in S$. Then by the principle of induction $S = \mathbb{N}$. This clearly implies $T = \mathbb{N}$.

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- Now by strong mathematical induction $T = \mathbb{N}$. This means that R is empty and we have a contradiction.

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- Suppose not. Take $T = \mathbb{N} \setminus R$.
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- Note. Here after we take it for granted that $\mathbb{N}$ has all these three properties.

Applications of Mathematical induction

- Suppose we have a property P defined for natural numbers, where (i) 1 satisfies property P; (ii) If $m \in \mathbb{N}$ satisfies property P then $(m+1)$ satisfies property P. Then property P is satisfied by all natural numbers.

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- Now using induction hypothesis

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- Hence $m+1 \in S$. Then by the principle of mathematical induction $S = \mathbb{N}$. In other words every natural number satisfies P.

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- So all the $m + 1$ balls are black. Quite Easily Done!

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- END OF LECTURE 4.