

# ANALYSIS -I

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## Lecture 5: Countable and Uncountable sets

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- ▶ We write  $A \sim B$  if  $B$  is equipotent with  $A$ .

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- ▶ This completes the proof that equipotency ( $\sim$ ) is an equivalence relation.

## Finite and infinite sets

- ▶ Definition 5.3: A set  $A$  is said to be **finite** if it is equipotent with  $\{1, 2, \dots, n\}$  for some  $n \in \mathbb{N}$  or it is empty. A set  $A$  is said to be **infinite** if it is not finite.

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- ▶ Even for infinite sets  $A, B$  we may informally say that  $A$  and  $B$  have same number of elements to mean that  $A$  and  $B$  are equipotent, even though we have not defined number of elements for infinite sets.

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- ▶ **Definition 5.6:** A set  $A$  is said to be **countable** if it is equipotent with  $\mathbb{N}$  or if it is finite. It is said to be **countably infinite** if is countable and not finite. A set  $A$  is said to be **uncountable** if it is not countable.

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- ▶ The manager can ask the new guest to take room number 1.

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- ▶ Then new guest  $h_n$  can go to room number  $(2n - 1)$  and we are done.

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- ▶ You may verify that  $h$  is a bijection.

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- ▶ Moral of the story: For infinite sets, a subset may have as many elements as the full set.

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- ▶ In other words for infinite sets disjoint union of sets of equal number of elements may again have same number of elements.

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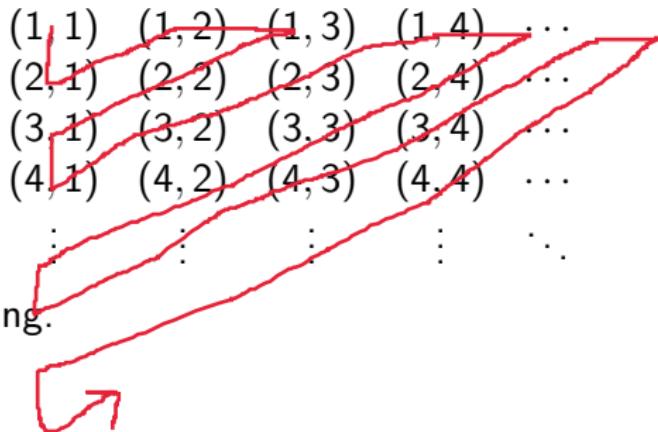
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- Zig-zag counting.
- We count the elements here as  
 $(1, 1), (2, 1), (1, 2), (1, 3), (2, 2), (3, 1), (4, 1), (3, 2), (2, 3), (1, 4), \dots$

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(4, 1)	(4, 2)	(4, 3)	(4, 4)	...
:	:	:	:	..

- Zig-zag counting.
- We count the elements here as  
 $(1, 1), (2, 1), (1, 2), (1, 3), (2, 2), (3, 1), (4, 1), (3, 2), (2, 3), (1, 4), \dots$
- This way we are able to exhaust all the elements of  $\mathbb{N} \times \mathbb{N}$ , without repeating any element twice.

## Cartesian product

- **Theorem 5.9:**  $\mathbb{N} \times \mathbb{N}$  is countable.
- **Proof:** Here is Cantor's argument.
- Look at  $\mathbb{N} \times \mathbb{N}$ .
- 

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- This way we are able to exhaust all the elements of  $\mathbb{N} \times \mathbb{N}$ , without repeating any element twice.
- In other words we have a bijection between  $\mathbb{N}$  and  $\mathbb{N} \times \mathbb{N}$ . In particular,  $\mathbb{N} \times \mathbb{N}$  is countable.

## Explicit bijections

► Exercise 5.10.1: Define  $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by

$$g(m, n) = 2^{m-1}(2n - 1), \quad (m, n) \in \mathbb{N} \times \mathbb{N}.$$

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- This problem is not very clearly stated. But we leave it at that.

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- ▶ **END OF LECTURE 5**