ANALYSIS -I

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 5: Countable and Uncountable sets

- Definition 5.1 Let A, B be two non-empty sets. Then B is said to be equipotent with A, if there exists a bijection $f : A \to B$. Empty set is equipotent to only itself.

Lecture 5: Countable and Uncountable sets

- Definition 5.1 Let A, B be two non-empty sets. Then B is said to be equipotent with A, if there exists a bijection $f : A \to B$. Empty set is equipotent to only itself.
- Some say B has same cardinality as A if B is equipotent with A.

Lecture 5: Countable and Uncountable sets

- Definition 5.1 Let A, B be two non-empty sets. Then B is said to be equipotent with A, if there exists a bijection $f : A \to B$. Empty set is equipotent to only itself.
- Some say B has same cardinality as A if B is equipotent with A.
- This means that B and A have 'same number of elements'. But currently we are not going to define 'cardinality' or number of elements for infinite sets. For this reason we prefer the terminology 'equipotent'.

Lecture 5: Countable and Uncountable sets

- Definition 5.1 Let A, B be two non-empty sets. Then B is said to be equipotent with A, if there exists a bijection $f : A \to B$. Empty set is equipotent to only itself.
- Some say B has same cardinality as A if B is equipotent with A.
- This means that B and A have 'same number of elements'. But currently we are not going to define 'cardinality' or number of elements for infinite sets. For this reason we prefer the terminology 'equipotent'.
- We write $A \sim B$ if B is equipotent with A.

Equivalence relation

- Theorem 5.2: Equipotency is an equivalence relation.

Equivalence relation

- Theorem 5.2: Equipotency is an equivalence relation.
- Proof: Claim 1: For any set A, $A \sim A$ (Reflexivity).

Equivalence relation

- Theorem 5.2: Equipotency is an equivalence relation.
- Proof: Claim 1: For any set A, $A \sim A$ (Reflexivity).
- If A is non-empty, we just take the identity function $i : A \to A$, defined by $i(a) = a$, $\forall a \in A$. If A is empty, $A \sim A$ by definition. This proves the claim.

Equivalence relation

- Theorem 5.2: Equipotency is an equivalence relation.
- Proof: Claim 1: For any set A, $A \sim A$ (Reflexivity).
- If A is non-empty, we just take the identity function $i : A \to A$, defined by $i(a) = a$, $\forall a \in A$. If A is empty, $A \sim A$ by definition. This proves the claim.
- Claim 2: If $A \sim B$ then $B \sim A$ (Symmetry).

Equivalence relation

- Theorem 5.2: Equipotency is an equivalence relation.
- Proof: Claim 1: For any set A, $A \sim A$ (Reflexivity).
- If A is non-empty, we just take the identity function $i : A \to A$, defined by $i(a) = a$, $\forall a \in A$. If A is empty, $A \sim A$ by definition. This proves the claim.
- Claim 2: If $A \sim B$ then $B \sim A$ (Symmetry).
- If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.

Equivalence relation

- Theorem 5.2: Equipotency is an equivalence relation.
- Proof: Claim 1: For any set A, $A \sim A$ (Reflexivity).
- If A is non-empty, we just take the identity function $i : A \to A$, defined by $i(a) = a$, $\forall a \in A$. If A is empty, $A \sim A$ by definition. This proves the claim.
- Claim 2: If $A \sim B$ then $B \sim A$ (Symmetry).
- If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.
- Indeed if $f^{-1}(x) = f^{-1}(y)$, then applying f, $x = y$. This shows that f^{-1} is injective.

Equivalence relation

- Theorem 5.2: Equipotency is an equivalence relation.
- Proof: Claim 1: For any set A, $A \sim A$ (Reflexivity).
- If A is non-empty, we just take the identity function $i : A \to A$, defined by $i(a) = a$, $\forall a \in A$. If A is empty, $A \sim A$ by definition. This proves the claim.
- Claim 2: If $A \sim B$ then $B \sim A$ (Symmetry).
- If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.
- Indeed if $f^{-1}(x) = f^{-1}(y)$, then applying f, $x = y$. This shows that f^{-1} is injective.
- If $a \in A$, then $a = f^{-1}(b)$, where $b = f(a)$. Hence f^{-1} is surjective. Combining the two statements, f^{-1} is bijective.

Equivalence relation

- Theorem 5.2: Equipotency is an equivalence relation.
- Proof: Claim 1: For any set A, $A \sim A$ (Reflexivity).
- If A is non-empty, we just take the identity function $i : A \to A$, defined by $i(a) = a$, $\forall a \in A$. If A is empty, $A \sim A$ by definition. This proves the claim.
- Claim 2: If $A \sim B$ then $B \sim A$ (Symmetry).
- If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.
- Indeed if $f^{-1}(x) = f^{-1}(y)$, then applying f, $x = y$. This shows that f^{-1} is injective.
- If $a \in A$, then $a = f^{-1}(b)$, where $b = f(a)$. Hence f^{-1} is surjective. Combining the two statements, f^{-1} is bijective.
- If A, B are empty then there is nothing to show.

Proof Continued

- Claim 3: Suppose $A \sim B$ and $B \sim C$, then $A \sim C$ (Transitivity).

Proof Continued

- Claim 3: Suppose $A \sim B$ and $B \sim C$, then $A \sim C$ (Transitivity).
- Suppose $f : A \to B$ is a bijection and $g : B \to C$ is a bijection.

Proof Continued

- Claim 3: Suppose $A \sim B$ and $B \sim C$, then $A \sim C$ (Transitivity).
- Suppose $f : A \to B$ is a bijection and $g : B \to C$ is a bijection.
- Then $h := g \circ f$ is a map from A to C.

Proof Continued

- Claim 3: Suppose $A \sim B$ and $B \sim C$, then $A \sim C$ (Transitivity).
- Suppose $f : A \to B$ is a bijection and $g : B \to C$ is a bijection.
- Then $h := g \circ f$ is a map from A to C.
- It is easy to see that h is a bijection.

Proof Continued

- Claim 3: Suppose $A \sim B$ and $B \sim C$, then $A \sim C$ (Transitivity).
- Suppose $f : A \to B$ is a bijection and $g : B \to C$ is a bijection.
- Then $h := g \circ f$ is a map from A to C.
- It is easy to see that h is a bijection.
- If A, B, C are empty, there is nothing to show.

Proof Continued

- Claim 3: Suppose $A \sim B$ and $B \sim C$, then $A \sim C$ (Transitivity).
- Suppose $f : A \to B$ is a bijection and $g : B \to C$ is a bijection.
- Then $h := g \circ f$ is a map from A to C.
- It is easy to see that h is a bijection.
- If A, B, C are empty, there is nothing to show.
- This completes the proof that equipotency ($\sim$) is an equivalence relation.

Finite and infinite sets

- Definition 5.3: A set A is said to be finite if it is equipotent with $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or it is empty. A set A is said to be infinite if it is not finite.

Finite and infinite sets

- Definition 5.3: A set A is said to be finite if it is equipotent with $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or it is empty. A set A is said to be infinite if it is not finite.
- From the pigeonhole principle, if A is equipotent with $\{1, 2, \ldots, m\}$ and with $\{1, 2, \ldots, n\}$ then $m = n$.

Finite and infinite sets

- Definition 5.3: A set A is said to be finite if it is equipotent with $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or it is empty. A set A is said to be infinite if it is not finite.
- From the pigeonhole principle, if A is equipotent with $\{1, 2, \ldots, m\}$ and with $\{1, 2, \ldots, n\}$ then $m = n$.
- This allows us to define the number of elements of a finite set A as n, if A is equipotent with $\{1, 2, \ldots n\}$. If A is empty then the number of elements A is defined to be zero.

Finite and infinite sets

- Definition 5.3: A set A is said to be finite if it is equipotent with $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or it is empty. A set A is said to be infinite if it is not finite.
- From the pigeonhole principle, if A is equipotent with $\{1, 2, \ldots, m\}$ and with $\{1, 2, \ldots, n\}$ then $m = n$.
- This allows us to define the number of elements of a finite set A as n, if A is equipotent with $\{1, 2, \ldots n\}$. If A is empty then the number of elements A is defined to be zero.
- Example 5.4: $A = \{a, b, c\}$ and $B = \{x, y, z\}$ have same number of elements, namely 3, as both of them are equipotent with $\{1, 2, 3\}$.

Finite and infinite sets

- Definition 5.3: A set A is said to be finite if it is equipotent with $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or it is empty. A set A is said to be infinite if it is not finite.
- From the pigeonhole principle, if A is equipotent with $\{1, 2, \ldots, m\}$ and with $\{1, 2, \ldots, n\}$ then $m = n$.
- This allows us to define the number of elements of a finite set A as n, if A is equipotent with $\{1, 2, \ldots n\}$. If A is empty then the number of elements A is defined to be zero.
- Example 5.4: $A = \{a, b, c\}$ and $B = \{x, y, z\}$ have same number of elements, namely 3, as both of them are equipotent with $\{1, 2, 3\}$.
- Even for infinite sets A, B we may informally say that A and B have same number of elements to mean that A and B are equipotent, even though we have not defined number of elements for infinite sets.

Countable sets

- Theorem 5.5: The set of natural numbers $\mathbb{N}$ is infinite:

Countable sets

- Theorem 5.5: The set of natural numbers $\mathbb{N}$ is infinite:
- Proof: Suppose $g : \mathbb{N} \to \{1, 2, \ldots n\}$ is a bijection for some $n \in \mathbb{N}$. In particular g is injective.

Countable sets

- Theorem 5.5: The set of natural numbers $\mathbb{N}$ is infinite:
- Proof: Suppose $g : \mathbb{N} \to \{1, 2, \ldots n\}$ is a bijection for some $n \in \mathbb{N}$. In particular g is injective.
- Taking any $m > n$ and restricting g to $\{1, 2, \ldots, m\}$ we get an injective map, as restriction of any injective map to a non-empty subset in the domain is injective. This contradicts pigeonhole principle. Hence $\mathbb{N}$ is infinite.

Countable sets

- Theorem 5.5: The set of natural numbers $\mathbb{N}$ is infinite:
- Proof: Suppose $g : \mathbb{N} \to \{1, 2, \ldots n\}$ is a bijection for some $n \in \mathbb{N}$. In particular g is injective.
- Taking any $m > n$ and restricting g to $\{1, 2, \ldots, m\}$ we get an injective map, as restriction of any injective map to a non-empty subset in the domain is injective. This contradicts pigeonhole principle. Hence $\mathbb{N}$ is infinite.
- Definition 5.6: A set A is said to be countable if it is equipotent with $\mathbb{N}$ or if it is finite. It is said to be countably infinite if is countable and not finite. A set A is said to be uncountable if it is not countable.

A story

- Hilbert's Hotel: Hilbert built a large hotel, which has a room with room number n for every natural number n.

A story

- Hilbert's Hotel: Hilbert built a large hotel, which has a room with room number n for every natural number n.
- Here are some great features of this hotel.

A story

- Hilbert's Hotel: Hilbert built a large hotel, which has a room with room number n for every natural number n.
- Here are some great features of this hotel.
- Flexibility: Suppose one day the hotel is houseful and a new guest arrives.

A story

- Hilbert's Hotel: Hilbert built a large hotel, which has a room with room number n for every natural number n.
- Here are some great features of this hotel.
- Flexibility: Suppose one day the hotel is houseful and a new guest arrives.
- The hotel manager need not send away the new guest.

A story

- Hilbert's Hotel: Hilbert built a large hotel, which has a room with room number n for every natural number n.
- Here are some great features of this hotel.
- Flexibility: Suppose one day the hotel is houseful and a new guest arrives.
- The hotel manager need not send away the new guest.
- The manager instructs the guest who is in room number 1 to move to room number 2, and the one in room number 2 to move to 3 and so on.

A story

- Hilbert's Hotel: Hilbert built a large hotel, which has a room with room number n for every natural number n.
- Here are some great features of this hotel.
- Flexibility: Suppose one day the hotel is houseful and a new guest arrives.
- The hotel manager need not send away the new guest.
- The manager instructs the guest who is in room number 1 to move to room number 2, and the one in room number 2 to move to 3 and so on.
- This way no old guest has been asked to vacate, still room number 1 is free.

A story

- Hilbert's Hotel: Hilbert built a large hotel, which has a room with room number n for every natural number n.
- Here are some great features of this hotel.
- Flexibility: Suppose one day the hotel is houseful and a new guest arrives.
- The hotel manager need not send away the new guest.
- The manager instructs the guest who is in room number 1 to move to room number 2, and the one in room number 2 to move to 3 and so on.
- This way no old guest has been asked to vacate, still room number 1 is free.
- The manager can ask the new guest to take room number 1.

More guests

- What if there are two new guests?

More guests

- What if there are two new guests?
- Well, either we can go through the previous procedure of accommodating one new person twice, or we can simply ask the present guest at room number n to go to room number $n + 2$ so that two rooms are freed up.

More guests

- What if there are two new guests?
- Well, either we can go through the previous procedure of accommodating one new person twice, or we can simply ask the present guest at room number n to go to room number $n + 2$ so that two rooms are freed up.
- What if there are infinitely many new guests? Say present guests are $g_1, g_2, \dots$ and new guests are $h_1, h_2, \dots,$.

More guests

- What if there are two new guests?
- Well, either we can go through the previous procedure of accommodating one new person twice, or we can simply ask the present guest at room number n to go to room number $n+2$ so that two rooms are freed up.
- What if there are infinitely many new guests? Say present guests are $g_1, g_2, \ldots$ and new guests are $h_1, h_2, \ldots,$.
- We can ask present guest g_n in room number n to go to room number $2n$, so that all odd numbered rooms are freed up.

More guests

- What if there are two new guests?
- Well, either we can go through the previous procedure of accommodating one new person twice, or we can simply ask the present guest at room number n to go to room number $n + 2$ so that two rooms are freed up.
- What if there are infinitely many new guests? Say present guests are $g_1, g_2, \ldots$ and new guests are $h_1, h_2, \ldots,$.
- We can ask present guest g_n in room number n to go to room number $2n$, so that all odd numbered rooms are freed up.
- Then new guest h_n can go to room number number $(2n - 1)$ and we are done.

Countably infinite sets

- Example 5.7: The set $\mathbb{N}_+ = \{0, 1, 2, \ldots\}$ is countable.

Countably infinite sets

- Example 5.7: The set $\mathbb{N}_+ = \{0, 1, 2, \ldots\}$ is countable.
- Indeed the function $g : \mathbb{N}_+ \to \mathbb{N}$ defined by

$$g(n) = n + 1, \quad \forall n \in \mathbb{N}_+$$

is easily seen to be a bijection.

Countably infinite sets

- Example 5.7: The set $\mathbb{N}_+ = \{0, 1, 2, \ldots\}$ is countable.
- Indeed the function $g : \mathbb{N}_+ \to \mathbb{N}$ defined by

$$g(n) = n + 1, \quad \forall n \in \mathbb{N}_+$$

  is easily seen to be a bijection.
- Example 5.8: The set $\mathbb{Z}$ of integers is countable:

Countably infinite sets

- Example 5.7: The set $\mathbb{N}_+ = \{0, 1, 2, \ldots\}$ is countable.
- Indeed the function $g : \mathbb{N}_+ \to \mathbb{N}$ defined by

$$g(n) = n + 1, \quad \forall n \in \mathbb{N}_+$$

  is easily seen to be a bijection.
- Example 5.8: The set $\mathbb{Z}$ of integers is countable:
- Define $h : \mathbb{Z} \to \mathbb{N}$ by

$$h(n) = \begin{cases} 2n & \text{if } \; n \geq 1 \\ -2n + 1 & \text{if } \; n \leq 0 \end{cases}$$

Countably infinite sets

- Example 5.7: The set $\mathbb{N}_+ = \{0, 1, 2, \ldots\}$ is countable.
- Indeed the function $g : \mathbb{N}_+ \to \mathbb{N}$ defined by

$$g(n) = n + 1, \quad \forall n \in \mathbb{N}_+$$

  is easily seen to be a bijection.
- Example 5.8: The set $\mathbb{Z}$ of integers is countable:
- Define $h : \mathbb{Z} \to \mathbb{N}$ by

$$h(n) = \begin{cases} 2n & \text{if } \ n \geq 1 \\ -2n+1 & \text{if } \ n \leq 0 \end{cases}$$

- You may verify that h is a bijection.

More or less

- There are more even numbers or more natural numbers?

More or less

- There are more even numbers or more natural numbers?
- $E = \{2, 4, 6, 8, \ldots\}$

More or less

- There are more even numbers or more natural numbers?
- $E = \{2, 4, 6, 8, \ldots\}$
- $\mathbb{N} = \{1, 2, 3, \ldots\}$.

More or less

- There are more even numbers or more natural numbers?
- $E = \{2, 4, 6, 8, \ldots\}$
- $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- On first look, it seems there are more natural numbers than even numbers.

More or less

- There are more even numbers or more natural numbers?
- $E = \{2, 4, 6, 8, \ldots\}$
- $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- On first look, it seems there are more natural numbers than even numbers.
- However, $g : \mathbb{N} \to E$ defined by $g(n) = 2n$ is a bijection. So there are as many even numbers as there are natural numbers. Not less! Note more!

More or less

- There are more even numbers or more natural numbers?
- $E = \{2, 4, 6, 8, \ldots\}$
- $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- On first look, it seems there are more natural numbers than even numbers.
- However, $g : \mathbb{N} \to E$ defined by $g(n) = 2n$ is a bijection. So there are as many even numbers as there are natural numbers. Not less! Note more!
- Moral of the story: For infinite sets, a subset may have as many elements as the full set.

Disjoint union

- Consider the set of odd natural numbers $H = \{1, 3, 5, \ldots\}$ and the set of even natural numbers $E = \{2, 4, 6, \ldots\}$.

Disjoint union

- Consider the set of odd natural numbers $H = \{1, 3, 5, \ldots\}$ and the set of even natural numbers $E = \{2, 4, 6, \ldots\}$.
- Now H, E have same number of elements and their union $\mathbb{N}$ also has same number of elements!

Disjoint union

- Consider the set of odd natural numbers $H = \{1, 3, 5, \dots\}$ and the set of even natural numbers $E = \{2, 4, 6, \dots\}$.
- Now H, E have same number of elements and their union $\mathbb{N}$ also has same number of elements!
- In other words for infinite sets disjoint union of sets of equal number of elements may again have same number of elements.

Cartesian product

- Theorem 5.9: $\mathbb{N} \times \mathbb{N}$ is countable.

Cartesian product

- Theorem 5.9: $\mathbb{N} \times \mathbb{N}$ is countable.
- Proof: Here is Cantor's argument.

Cartesian product

- Theorem 5.9: $\mathbb{N} \times \mathbb{N}$ is countable.
- Proof: Here is Cantor's argument.
- Look at $\mathbb{N} \times \mathbb{N}$.

Cartesian product

- Theorem 5.9: $\mathbb{N} \times \mathbb{N}$ is countable.
- Proof: Here is Cantor's argument.
- Look at $\mathbb{N} \times \mathbb{N}$.
- $$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Cartesian product

- Theorem 5.9: $\mathbb{N} \times \mathbb{N}$ is countable.
- Proof: Here is Cantor's argument.
- Look at $\mathbb{N} \times \mathbb{N}$.
-

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

- Zig-zag counting.

Cartesian product

- Theorem 5.9: $\mathbb{N} \times \mathbb{N}$ is countable.
- Proof: Here is Cantor's argument.
- Look at $\mathbb{N} \times \mathbb{N}$.
- $$\begin{array}{ccccc} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$
- Zig-zag counting.
- We count the elements here as $(1,1),(2,1),(1,2),(1,3),(2,2),(3,1),(4,1),(3,2),(2,3),(1,4),\ldots,$

Cartesian product

- Theorem 5.9: $\mathbb{N} \times \mathbb{N}$ is countable.
- Proof: Here is Cantor's argument.
- Look at $\mathbb{N} \times \mathbb{N}$.
- $$\begin{array}{ccccc} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$
- Zig-zag counting.
- We count the elements here as $(1,1),(2,1),(1,2),(1,3),(2,2),(3,1),(4,1),(3,2),(2,3),(1,4),\ldots,$
- This way we are able to exhaust all the elements of $\mathbb{N} \times \mathbb{N}$, without repeating any element twice.

Cartesian product

- Theorem 5.9: $\mathbb{N} \times \mathbb{N}$ is countable.
- Proof: Here is Cantor's argument.
- Look at $\mathbb{N} \times \mathbb{N}$.
- $$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$
- Zig-zag counting.
- We count the elements here as $(1,1),(2,1),(1,2),(1,3),(2,2),(3,1),(4,1),(3,2),(2,3),(1,4),\ldots,$
- This way we are able to exhaust all the elements of $\mathbb{N} \times \mathbb{N}$, without repeating any element twice.
- In other words we have a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. In particular, $\mathbb{N} \times \mathbb{N}$ is countable.

Explicit bijections

- Exercise 5.10.1: Define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(m, n) = 2^{m-1}(2n - 1), \quad (m, n) \in \mathbb{N} \times \mathbb{N}.$$

Explicit bijections

- Exercise 5.10.1: Define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(m, n) = 2^{m-1}(2n - 1), \quad (m, n) \in \mathbb{N} \times \mathbb{N}.$$

- Show that g is a bijection.

Explicit bijections

- Exercise 5.10.1: Define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(m,n) = 2^{m-1}(2n-1), \quad (m,n) \in \mathbb{N} \times \mathbb{N}.$$

- Show that g is a bijection.
- Exercise 5.10.2: Define $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$h(m,n) = m + [\frac{(m+n-1)(m+n-2)}{2}], \quad (m,n) \in \mathbb{N} \times \mathbb{N}.$$

Explicit bijections

- Exercise 5.10.1: Define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(m, n) = 2^{m-1}(2n - 1), \quad (m, n) \in \mathbb{N} \times \mathbb{N}.$$

- Show that g is a bijection.
- Exercise 5.10.2: Define $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$h(m, n) = m + [\frac{(m + n - 1)(m + n - 2)}{2}], \quad (m, n) \in \mathbb{N} \times \mathbb{N}.$$

- Show that h is a bijection.

Explicit bijections

- Exercise 5.10.1: Define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(m,n) = 2^{m-1}(2n-1), \quad (m,n) \in \mathbb{N} \times \mathbb{N}.$$

- Show that g is a bijection.
- Exercise 5.10.2: Define $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$h(m,n) = m + [\frac{(m+n-1)(m+n-2)}{2}], \quad (m,n) \in \mathbb{N} \times \mathbb{N}.$$

- Show that h is a bijection.
- Challenge Problem 3: Obtain another 'explicit' bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ different from $g, h, \tilde{g}, \tilde{h}$, where $\tilde{g}(m,n) = g(n,m)$, and $\tilde{h}(m,n) = h(n,m), \quad \forall m,n \in \mathbb{N} \times \mathbb{N}$.

Explicit bijections

- Exercise 5.10.1: Define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(m,n) = 2^{m-1}(2n-1), \quad (m,n) \in \mathbb{N} \times \mathbb{N}.$$

- Show that g is a bijection.
- Exercise 5.10.2: Define $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$h(m,n) = m + [\frac{(m+n-1)(m+n-2)}{2}], \quad (m,n) \in \mathbb{N} \times \mathbb{N}.$$

- Show that h is a bijection.
- Challenge Problem 3: Obtain another 'explicit' bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ different from $g, h, \tilde{g}, \tilde{h}$, where $\tilde{g}(m,n) = g(n,m)$, and $\tilde{h}(m,n) = h(n,m), \quad \forall m,n \in \mathbb{N} \times \mathbb{N}$.
- This problem is not very clearly stated. But we leave it at that.

Schroder-Bernstein theorem

- The pigeon hole principle suggests that if we have an injective function $f : A \to B$, then B should be having 'more' elements than A.

Schroder-Bernstein theorem

- The pigeon hole principle suggests that if we have an injective function $f : A \rightarrow B$, then B should be having 'more' elements than A.
- What if there is an injective function from A to B and another injective function from B to A?

Schroder-Bernstein theorem

- The pigeon hole principle suggests that if we have an injective function $f : A \to B$, then B should be having 'more' elements than A.
- What if there is an injective function from A to B and another injective function from B to A?
- Theorem 5.11 (Schroder-Bernstein theorem): Let A, B be non-empty sets. Suppose there exist injective functions $f : A \to B$ and $g : B \to A$. Then there exists a bijective function $h : A \to B$. Consequently A and B are equipotent.

Schroder-Bernstein theorem

- The pigeon hole principle suggests that if we have an injective function $f : A \to B$, then B should be having 'more' elements than A.
- What if there is an injective function from A to B and another injective function from B to A?
- Theorem 5.11 (Schroder-Bernstein theorem): Let A, B be non-empty sets. Suppose there exist injective functions $f : A \to B$ and $g : B \to A$. Then there exists a bijective function $h : A \to B$. Consequently A and B are equipotent.
- Exercise 5.12: Prove Schroder-Bernstein theorem. If you are unable to prove it yourself, discuss with your friends. Still if you can't do it, get a proof from the internet and understand it!

Schroder-Bernstein theorem

- The pigeon hole principle suggests that if we have an injective function $f : A \to B$, then B should be having 'more' elements than A.
- What if there is an injective function from A to B and another injective function from B to A?
- Theorem 5.11 (Schroder-Bernstein theorem): Let A, B be non-empty sets. Suppose there exist injective functions $f : A \to B$ and $g : B \to A$. Then there exists a bijective function $h : A \to B$. Consequently A and B are equipotent.
- Exercise 5.12: Prove Schroder-Bernstein theorem. If you are unable to prove it yourself, discuss with your friends. Still if you can't do it, get a proof from the internet and understand it!
- END OF LECTURE 5