ANALYSIS -I

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- Definition 5.3: A set A is said to be finite if it is equipotent with $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or it is empty. A set A is said to be infinite if it is not finite.
- Definition 5.6: A set A is said to be countable if it is equipotent with $\mathbb{N}$ or if it is finite. It is said to be countably infinite if is countable and not finite. A set A is said to be uncountable if it is not countable.

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- We saw that $\mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N}$ are all countable.
- Now it is time to see some uncountable sets.

Binary sequences

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- Proof: Suppose that there exists a bijection $f : \mathbb{N} \to \mathbb{B}$. In particular f is a surjection.
- Then for every $i \in \mathbb{N}$, $f(i)$ is a binary sequence.

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- Look at the infinite matrix:

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- formed by writing down $f(1), f(2), \ldots$ as rows.
- Form a binary sequence using the diagonal entries: $(w_{11}, w_{22}, w_{33}, \ldots)$.
- We flip the entries to get a new binary sequence, $v = (v_1, v_2, v_3, \ldots)$ where $v_j = 1 - w_{jj}$ for every $j \in \mathbb{N}$. Now we claim that v is not in the range of f.

Proof Continued

- $v \neq f(1)$ as $v = (v_1, v_2, \ldots)$, $f(1) = (w_{11}, w_{12}, \ldots)$ and $v_1 = 1 - w_{11} \neq w_{11}$. So the first entry does not match.

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- In particular $\mathbb{B}$ is not countable.

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- Exercise: If A is a finite set with n elements, show that $P(A)$ has 2^n elements.
- We guess that $P(A)$ should be having ‘more’ elements than A.

Power sets -continued

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- Clearly D is a subset of A, and hence it is an element of $P(A)$.
- We claim that D is not in the range of f. That would show that f is not surjective.

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- But $f(a_0) = D$. Hence $a_0 \notin D$. This contradicts $a_0 \in D$.

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- Therefore our assumption that D is in the range of f must be wrong. Consequently f is not surjective.

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- In other words, $c(j) := c_j$, is just the 'indicator function' of the set C.
- Now go back and see that the proof of last theorem and that of uncountability of $\mathbb{B}$ use the same idea!

Bigger and bigger infinities

- We have seen that $P(\mathbb{N})$ is bigger than $\mathbb{N}$ in the sense that there is no surjective function from $\mathbb{N}$ to $P(\mathbb{N})$. [There are of course, surjective functions from $P(\mathbb{N})$ to $\mathbb{N}$. (Why?).]

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- So there are bigger and bigger infinities.

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- More generally, if A, B are non-empty finite sets, A has n elements and B has m elements, then B^A has m^n elements.

Spaces of functions

- Let A, B be non-empty sets. Let B^A denote the set of all functions from A to B.
- For $n \in \mathbb{N}$, if $A = \{1, 2, \ldots, n\}$ and $B = \{0, 1\}$, then observe that B^A has 2^n elements.
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- More generally, if A, B are non-empty finite sets, A has n elements and B has m elements, then B^A has m^n elements.
- Observe that for any non-empty set A, if $B = \{0, 1\}$ then B^A is equipotent with the power set of A.
- Observe that $B^{\mathbb{N}}$ is same as the space of sequences with elements from B. In particular, if $B = \{0, 1\}$, then $B^{\mathbb{N}}$ is same as the space of binary sequences.

Hilbert's hotel

- Link 1:

  https : //youtu.be/OxGsU8oIWjY

Hilbert's hotel

- Link 1:

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- Link 2:

  $$https://youtu.be/Uj3_KqkI9Zo$$

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- END OF LECTURE 6