ANALYSIS -I

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Lecture 7: Real Numbers

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- For instance, we construct positive rational numbers out of $\mathbb{N} \times \mathbb{N}$, by identifying (a, b) with (a', b') if $ab' = a'b$. (Think of (a, b) as $\frac{a}{b}$.) However, we will not take such an approach.
- If you wish, you may see the construction of real numbers in due course once you are fully familiar with various properties of real numbers.

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$$a + (b + c) = (a + b) + c, \quad \forall a, b, c \in \mathbb{R}.$$

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Addition Axioms continued

- A3. There exists an element called ‘zero’, denoted by ‘0’ in $\mathbb{R}$ such that
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- A4. For every $a \in \mathbb{R}$, there exists an element '$-a$' in $\mathbb{R}$ such that
$$a + (-a) = (-a) + a = 0.$$
-Existence of additive inverse. $-a$ is known as additive inverse of a.

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- Note that we have explicitly assumed that $1 \neq 0$

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- This axiom binds addition and multiplication.

Consequences

- Theorem 7.1 : (i) (Uniqueness of 0). If $e \in \mathbb{R}$ satisfies $a + e = e + a = a$ for all $a \in \mathbb{R}$, then $e = 0$. (ii) (uniqueness of 1). If $f \in \mathbb{R}$ satisfies $a.f = f.a = a$ for all $a \in \mathbb{R}$, then $f = 1$.

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- Proof: (i) Take $a = 0$. Then we get $0 + e = e + 0 = 0$. But by A3, $0 + e = e + 0 = e$. Hence $e = 0$. (ii)Take $a = 1$ and we get $1.f = f.1 = 1$ and also $1.f = f.1 = f$. Hence $f = 1$.

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- Proof: This is clear from the cancellation property of addition, as $a + a_1 = a + (-a)$.

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- (iii) Given $a, b \in \mathbb{R}$ and $a.b = 0$.
- Now suppose $a \neq 0$, then a^{-1} exists and we get

$$a^{-1}.(a.b) = a^{-1}.0 = 0.$$

Hence by associativity of multiplication, $(a^{-1}.a).b = 0$, or $1.b = 0$, which implies $b = 0$. So either $a = 0$ or $b = 0$.

Natural numbers

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- You may verify that all natural numbers are distinct.

Integers, rational numbers and irrational numbers

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- END OF LECTURE 7.