ANALYSIS -I

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Lecture 8: Real Numbers : Order axioms

- We are assuming that there is a set called set of real numbers $\mathbb{R}$ with two binary operations', $+$, $.$, satisfying certain axioms.

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- A4. For every $a \in \mathbb{R}$, there exists an element '$-a$' in $\mathbb{R}$ such that
$$a + (-a) = (-a) + a = 0.$$
-Existence of additive inverse. $-a$ is known as additive inverse of a.

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- These axioms are known as algebraic axioms. They determine the 'algebraic structure' of real numbers.

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- O3. If $a \in \mathbb{R}$, then exactly one of the following three properties is true:
  (i) $a \in \mathbb{P}$;
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  [This is known as trichotomy property for real numbers.]
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- Any element of $\mathbb{P}$ is said to be positive.
- Warning: The notation $\mathbb{P}$ for positive real numbers is not standard. You may see $\mathbb{R}^+$, $(0, \infty)$ as some of the alternative notations for the set of positive real numbers.

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- Consider the set S of all natural numbers which are positive. Then $1 \in S$ and if $n \in S$, then $n + 1 \in S$.
- Now a simple application of mathematical induction shows that $n \in \mathbb{P}$ for every $n \in \mathbb{N}$.

Inequalities

- Notation: For real numbers, a, b, we write $a < b$ or $b > a$ if $b - a \in \mathbb{P}$. We write $a \leq b$ or $b \geq a$ if $b - a \in \mathbb{P} \bigcup \{0\}$.

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- Now order axioms under this notation, becomes:
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- We may call a real number a as negative if $-a$ is positive.

Simple inequalities

- Theorem 8.2: Suppose a, b, c, d are real numbers. Then
  (i) If $a < b$, then $a + c < b + c$.
  (ii) If $a \leq b$, then $a + c \leq b + c$.
  (iii) If $a < b$ and $c < d$, then $a + c < b + d$.
  (iv) If $a < b$ and $c > 0$, then $ac < bc$.
  (v) If $a < b$ and $c < 0$, then $a > b$.
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- Often we show two real numbers a, b are equal by showing $a \leq b$ and $b \leq a$. The equality follows by trichotomy property.

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- Proof. Suppose $a < b$. Now $b^2 - a^2 = (b + a)(b - a)$. As, both $(b + a)$ and $(b - a)$ are positive, $b^2 - a^2$ is positive. In other words, $a^2 < b^2$.

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- Conversely, suppose $a^2 < b^2$. Hence $(b^2 - a^2) = (b + a)(b - a)$ is positive. As a, b are assumed to be positive, $(b + a)$ is positive. Now from Theorem 8.1 it is clear that for the product $(b + a)(b - a)$ to be positive, we also need $(b - a)$ positive.

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- Now if a is positive and b is negative, say $b = -|b|$, with $0 < |b| \leq a$, we get
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$|a + b| = |a - |b|| = a - |b| \leq a = |a| \leq |a| + |b|$.
- Similarly if a is positive and b is negative with $0 < a \leq |b|$, we get $|a + b| = |a - |b|| = |b| - a \leq |b| \leq |a| + |b|$. Other cases

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- You will see that this notion of distance has far reaching applications in Analysis.

No smallest or largest positive elements

- Theorem 8.5: (i) The set $\mathbb{P}$ has no least element, that is, there exists no positive real number α, such that $\alpha \leq a$ for every positive real number a. (ii) The set $\mathbb{P}$ has no largest element, that is, there exists no positive real number β, such that $a \leq \beta$ for every positive real number a.

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- Proof: Suppose α is a positive real number. Then we claim $0 < \frac{\alpha}{2} < \alpha$.
- It is easy to see that $2^{-1} = \frac{1}{2}$ is positive (Otherwise $1 = 2.2^{-1}$ would be negative). Hence $\frac{\alpha}{2} = \alpha.\frac{1}{2}$ is positive.

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- It is easy to see that $2^{-1} = \frac{1}{2}$ is positive (Otherwise $1 = 2.2^{-1}$ would be negative). Hence $\frac{\alpha}{2} = \alpha.\frac{1}{2}$ is positive.
- So $\alpha - \frac{\alpha}{2} = \frac{\alpha}{2}$ is also positive.

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- END OF LECTURE 8.