ANALYSIS -I

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Lecture 1: Introduction

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- Let me know your interests.
- Mathematical as well as non-mathematical.
- Tell me which result you like most!

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My take

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- We do logical thinking.
- Given a set of statements, what are the statements we can deduce is what bothers us most of the time.
- We learn to make these deductions systematically.
- The statements we start with or which we take for granted are axioms.
- We think of some deductions as important or beautiful. We call them as theorems.

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- There is no contradiction here! Non-smoking also may cause cancer!
- Starting with a small set of axioms, the whole edifice of mathematics is built using logical deductions.

Applications

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- So on.
- We see structural, logical similarities in many different contexts.

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- Keeping the information safe is done using cryptology. That also uses mathematics in a non-trivial way.

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- The setting should be clear. The statements should be clear, the deductions should be clear and so on.

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- An engineer, a physicist, and a mathematician were on a train heading north, and had just crossed the border into Scotland.
- The engineer looked out of the window and said "Look! Scottish sheep are black!"
- The physicist said, "No, no. Some Scottish sheep are black."
- The mathematician looked irritated and said: "All we can say is that there is one field, containing at least one sheep, of which at least one side is black, as of now."

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- In other words all these topics are deeply inter-connected. Simply said, mathematics is one subject.

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- In other words all these topics are deeply inter-connected. Simply said, mathematics is one subject.
- You should learn basics of all the areas for now. Specialization comes only at an advanced level. You should not bother about it for now. Just have an open mind about all the areas.

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- In other words, there exists an element j which is contained in at least half the sets in $\mathcal{F}$.

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- $\mathcal{F}_3 = \{A \subseteq S : 1 \notin A\}$.
- $\mathcal{F}_4 = \{A \subseteq S : \sharp A = 2\}$.
- Then $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ satisfy conditions (i), (ii). $\mathcal{F}_4$ does not satisfy condition (iii).

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- $\mathcal{F}_1 = \{\emptyset, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$.
- $\sharp\mathcal{F}_1 = 4$; and we can take $j = 2$. There are three sets in $\mathcal{F}_1$ containing j.

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- END OF LECTURE 1.

Lecture 2: Set theory and Russell's paradox

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- $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$-the set of integers.

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- The main point here is that given an object we should be clear as to whether it is an element of the set or not.
- This is a requirement so that we do not have any confusion. Still the definition is only an informal one.

Russell's barber paradox

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- Let us see some more paradoxes of similar type.

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- What about the adjective 'HETEROLOGICAL? We again face a problem.

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- Let A be the set of all ordinary sets and let B be the set of all extraordinary sets.
- Is A ordinary or extraordinary? Either way we have a problem!

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- You may see details ZFC later on if you do a course on

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- In the usual picture of graphs of functions on real line this is known as vertical line test. A graph of a function can not be touching a vertical line at more than one point.

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- Sometimes people call B, the co-domain as range of f. It is better to avoid that kind of terminology as it can lead to confusion.

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- It is fine, if some rooms are vacant. In other words, there could be $y \in B$ such that $y \neq f(x)$ for any $x \in A$.
- It is also fine if students are asked to share rooms. In other words it is possible to have x, x' in A, such that $f(x) = f(x')$.

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- While allotting rooms to students, injectivity or one-to-one means there is no sharing of rooms.

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- In the problem of allotting rooms to students it means that the hostel is full. That is all the rooms have got allotted.

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- Define $f_2 : \mathbb{Z} \to \mathbb{Z}$ by $f_2(n) = -n, \quad \forall n \in \mathbb{Z}$. Then f_2 is a bijection.
- Define $f_3 : \mathbb{Z} \to \mathbb{Z}$ by $f_3(n) = n^2$. Then f_3 is neither injective nor surjective.

Compositions of functions

- Let A, B, C be non-empty sets. Let $f : A \to B$ and $g : B \to C$ be functions. Then a new function $g \circ f : A \to C$ is got by taking
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- The out put of machine f is taken as input for g.

Inverse map

- Let A, B be non-empty sets and let $f : A \to B$ be a bijection. Then we see that for every $b \in B$ there exists unique $a \in A$ such that $f(a) = b$. Then we call a as $f^{-1}(b)$.

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- The identity map is a completely lazy machine where the output is same as the input.

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One sided inverses

- Example: Suppose $A = \{x, y\}$ and $B = \{4, 5, 6\}$.
- Define $f : A \to B$ by $f(x) = 4$ and $f(y) = 6$.
- Define $g : B \to A$ by $g(4) = g(5) = x$ and $g(6) = y$.
- Then $g \circ f(x) = x$ and $g \circ f(y) = y$.
- So $g \circ f$ is the identity map on A. However, $f \circ g$ is not the identity map on B.

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- Proof: Take $h = g \circ f$. Suppose $f(a_1) = f(a_2)$ for some a_1, a_2 in A. Then by the definition of a function, $g(f(a_1)) = g(f(a_2))$. In other words, $h(a_1) = h(a_2)$. But h is assumed to be one to one. Hence $a_1 = a_2$. This shows that f is one to one.

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- Proof: Exercise!

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- Similarly $f^3(a) = (f \circ f \circ f)(a) = f(f(f(a)))$.
- More generally, we can define f^n for any natural number n.
- Note that in general you can not define f^2 when f is a function from one set to a different set.

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- Consider $h : \mathbb{Z} \to \mathbb{Z}$ defined by

$$h(n) = \begin{cases} 3k & \text{if} \quad n = 2k, \quad k \in \mathbb{Z} \\ 3k+1 & \text{if} \quad n = 4k+1 \quad k \in \mathbb{Z} \\ 3k-1 & \text{if} \quad n = 4k-1 \quad k \in \mathbb{Z} \end{cases}$$

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- END OF LECTURE 3.

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- Let us look at a few basic properties of the set of natural numbers and its subsets.

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- Note that clearly the minimal element of R is unique, for if both k, l are minimal then we have $k \leq l$ and $l \leq k$, and this means $k = l$.
- We also note that if $n \in R$, then the minimal element of R is contained in $\{1, 2, \ldots, n\} \bigcap R$. So the existence of minimum here is essentially a statement about finite sets.

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- Note. Here after we take it for granted that $\mathbb{N}$ has all these three properties.

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- Suppose we have a property P defined for natural numbers, where (i) 1 satisfies property P; (ii) If $m \in \mathbb{N}$ satisfies property P then $(m+1)$ satisfies property P. Then property P is satisfied by all natural numbers.

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- Hence $m+1 \in S$. Then by the principle of mathematical induction $S = \mathbb{N}$. In other words every natural number satisfies P.

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- Now there are m balls in the bag, and all of them have the same color, say black, by the induction hypothesis.
- Now put the ball you have in hand in bag and remove some other. Clearly the ball you have removed must be black color. Consider the balls in the bag. Now there are only m of them, also have to be of same color, same as the one ball we removed.

A fake theorem

- "Theorem": If you take bag full of balls all of them would have same color.
- "Proof":" We will prove this by induction.
- Let n be the number of balls in the bag.
- If $n = 1$, the claim is obvious. There is nothing to prove.
- Now assume the result for $n = m$ and we will prove it for $n = m + 1$.
- Suppose the bag has $m + 1$ balls. Remove one ball.
- Now there are m balls in the bag, and all of them have the same color, say black, by the induction hypothesis.
- Now put the ball you have in hand in bag and remove some other. Clearly the ball you have removed must be black color. Consider the balls in the bag. Now there are only m of them, also have to be of same color, same as the one ball we removed.
- So all the $m + 1$ balls are black. Quite Easily Done!

Pigeonhole principle

- Pigeonhole principle: Let m, n be natural numbers and $m < n$. Let
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- END OF LECTURE 4.

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- We write $A \sim B$ if B is equipotent with A.

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- This completes the proof that equipotency ($\sim$) is an equivalence relation.

Finite and infinite sets

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- Example 5.4: $A = \{a, b, c\}$ and $B = \{x, y, z\}$ have same number of elements, namely 3, as both of them are equipotent with $\{1, 2, 3\}$.
- Even for infinite sets A, B we may informally say that A and B have same number of elements to mean that A and B are equipotent, even though we have not defined number of elements for infinite sets.

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- Definition 5.6: A set A is said to be countable if it is equipotent with $\mathbb{N}$ or if it is finite. It is said to be countably infinite if is countable and not finite. A set A is said to be uncountable if it is not countable.

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- Then new guest h_n can go to room number number $(2n - 1)$ and we are done.

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- You may verify that h is a bijection.

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- Moral of the story: For infinite sets, a subset may have as many elements as the full set.

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- In other words for infinite sets disjoint union of sets of equal number of elements may again have same number of elements.

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- This way we are able to exhaust all the elements of $\mathbb{N} \times \mathbb{N}$, without repeating any element twice.

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- $$\begin{array}{ccccc} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$
- Zig-zag counting.
- We count the elements here as $(1,1),(2,1),(1,2),(1,3),(2,2),(3,1),(4,1),(3,2),(2,3),(1,4),\ldots,$
- This way we are able to exhaust all the elements of $\mathbb{N} \times \mathbb{N}$, without repeating any element twice.
- In other words we have a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. In particular, $\mathbb{N} \times \mathbb{N}$ is countable.

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- Challenge Problem 3: Obtain another 'explicit' bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ different from $g, h, \tilde{g}, \tilde{h}$, where $\tilde{g}(m,n) = g(n,m)$, and $\tilde{h}(m,n) = h(n,m), \quad \forall m,n \in \mathbb{N} \times \mathbb{N}$.

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- This problem is not very clearly stated. But we leave it at that.

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- Now it is time to see some uncountable sets.

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- Proof: Suppose that there exists a bijection $f : \mathbb{N} \to \mathbb{B}$. In particular f is a surjection.
- Then for every $i \in \mathbb{N}$, $f(i)$ is a binary sequence.

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- Look at the infinite matrix:

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- Form a binary sequence using the diagonal entries: $(w_{11}, w_{22}, w_{33}, \dots)$.
- We flip the entries to get a new binary sequence, $v = (v_1, v_2, v_3, \dots)$ where $v_j = 1 - w_{jj}$ for every $j \in \mathbb{N}$. Now we claim that v is not in the range of f.

Proof Continued

- $v \neq f(1)$ as $v = (v_1, v_2, \ldots)$, $f(1) = (w_{11}, w_{12}, \ldots)$ and $v_1 = 1 - w_{11} \neq w_{11}$. So the first entry does not match.

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- Clearly D is a subset of A, and hence it is an element of $P(A)$.
- We claim that D is not in the range of f. That would show that f is not surjective.

Proof continued

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- So $D = f(a_0)$ for some $a_0 \in A$.
- Now either $a_0 \in D$ or $a_0 \notin D$.

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- Therefore our assumption that D is in the range of f must be wrong. Consequently f is not surjective.

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- In other words, $c(j) := c_j$, is just the 'indicator function' of the set C.
- Now go back and see that the proof of last theorem and that of uncountability of $\mathbb{B}$ use the same idea!

Bigger and bigger infinities

- We have seen that $P(\mathbb{N})$ is bigger than $\mathbb{N}$ in the sense that there is no surjective function from $\mathbb{N}$ to $P(\mathbb{N})$. [There are of course, surjective functions from $P(\mathbb{N})$ to $\mathbb{N}$. (Why?).]

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- Observe that $B^{\mathbb{N}}$ is same as the space of sequences with elements from B. In particular, if $B = \{0, 1\}$, then $B^{\mathbb{N}}$ is same as the space of binary sequences.

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- END OF LECTURE 6

Lecture 7: Real Numbers

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- If you wish, you may see the construction of real numbers in due course once you are fully familiar with various properties of real numbers.

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- A3. There exists an element called 'zero', denoted by '0' in $\mathbb{R}$ such that

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- Note that we have explicitly assumed that $1 \neq 0$

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- This axiom binds addition and multiplication.

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- Theorem 7.1 : (i) (Uniqueness of 0). If $e \in \mathbb{R}$ satisfies $a + e = e + a = a$ for all $a \in \mathbb{R}$, then $e = 0$. (ii) (uniqueness of 1). If $f \in \mathbb{R}$ satisfies $a.f = f.a = a$ for all $a \in \mathbb{R}$, then $f = 1$.

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- Theorem 7.4 (Cancellation property of multiplication): For $a, b, c \in \mathbb{R}$ with $a \neq 0$, if $a.b = a.c$ then $b = c$.

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- (iii) Given $a, b \in \mathbb{R}$ and $a.b = 0$.
- Now suppose $a \neq 0$, then a^{-1} exists and we get

$$a^{-1}.(a.b) = a^{-1}.0 = 0.$$

Hence by associativity of multiplication, $(a^{-1}.a).b = 0$, or $1.b = 0$, which implies $b = 0$. So either $a = 0$ or $b = 0$.

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- More generally, $n \in \mathbb{N}$ is identified with $1 + 1 + \cdots + 1(n \ \text{ times})$.
- You may verify that all natural numbers are distinct.

Integers, rational numbers and irrational numbers

- $\mathbb{Z}$ is also thought of as a subset of $\mathbb{R}$: $0 \in \mathbb{Z}$ is identified with 0 of $\mathbb{R}$ and $-n$ for $n \in \mathbb{N}$ is just the additive inverse of n.

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- END OF LECTURE 7.

Lecture 8: Real Numbers : Order axioms

- We are assuming that there is a set called set of real numbers $\mathbb{R}$ with two binary operations', $+$, $.$, satisfying certain axioms.

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$$a + 0 = 0 + a = a, \quad \forall a \in \mathbb{R}.$$
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- A4. For every $a \in \mathbb{R}$, there exists an element '$-a$' in $\mathbb{R}$ such that
$$a + (-a) = (-a) + a = 0.$$
-Existence of additive inverse. $-a$ is known as additive inverse of a.

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- M4. For every $a \in \mathbb{R}$, with $a \neq 0$, there exists an element 'a^{-1}' in $\mathbb{R}$ such that

$$a.a^{-1} = a^{-1}.a = 1.$$

-Existence of multiplicative inverse. a^{-1} is known as multiplicative inverse of a.

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- D. For a, b, c in $\mathbb{R}$,

$$a.(b+c) = a.b + a.c$$

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- These axioms are known as algebraic axioms. They determine the 'algebraic structure' of real numbers.

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- O3. If $a \in \mathbb{R}$, then exactly one of the following three properties is true:
  (i) $a \in \mathbb{P}$;
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  [This is known as trichotomy property for real numbers.]
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- Warning: The notation $\mathbb{P}$ for positive real numbers is not standard. You may see $\mathbb{R}^+$, $(0, \infty)$ as some of the alternative notations for the set of positive real numbers.

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- Consider the set S of all natural numbers which are positive. Then $1 \in S$ and if $n \in S$, then $n + 1 \in S$.
- Now a simple application of mathematical induction shows that $n \in \mathbb{P}$ for every $n \in \mathbb{N}$.

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- Notation: For real numbers, a, b, we write $a < b$ or $b > a$ if $b - a \in \mathbb{P}$. We write $a \leq b$ or $b \geq a$ if $b - a \in \mathbb{P} \bigcup \{0\}$.

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- We may call a real number a as negative if $-a$ is positive.

Simple inequalities

- Theorem 8.2: Suppose a, b, c, d are real numbers. Then
  (i) If $a < b$, then $a + c < b + c$.
  (ii) If $a \leq b$, then $a + c \leq b + c$.
  (iii) If $a < b$ and $c < d$, then $a + c < b + d$.
  (iv) If $a < b$ and $c > 0$, then $ac < bc$.
  (v) If $a < b$ and $c < 0$, then $a > b$.
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- Often we show two real numbers a, b are equal by showing $a \leq b$ and $b \leq a$. The equality follows by trichotomy property.

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- Theorem 8.3: If a, b are positive real numbers, then $a^2 < b^2$ if and only if $a < b$.
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- Conversely, suppose $a^2 < b^2$. Hence $(b^2 - a^2) = (b + a)(b - a)$ is positive. As a, b are assumed to be positive, $(b + a)$ is positive. Now from Theorem 8.1 it is clear that for the product $(b + a)(b - a)$ to be positive, we also need $(b - a)$ positive.

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- For any real number a, the modulus of a, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \text{if} \;\; a \geq 0; \\ -a & \text{if} \;\; a < 0. \end{cases}$$

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- Now if a is positive and b is negative, say $b = -|b|$, with $0 < |b| \leq a$, we get $|a + b| = |a - |b|| = a - |b| \leq a = |a| \leq |a| + |b|$.

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- Similarly if a is positive and b is negative with $0 < a \leq |b|$, we get $|a + b| = |a - |b|| = |b| - a \leq |b| \leq |a| + |b|$. Other cases

Why is this triangle inequality?

- Suppose a, b are any two real numbers. Define the 'distance' between a and b as

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- Now it should be clear as to why this is called triangle inequality.
- You will see that this notion of distance has far reaching applications in Analysis.

No smallest or largest positive elements

- Theorem 8.5: (i) The set $\mathbb{P}$ has no least element, that is, there exists no positive real number α, such that $\alpha \leq a$ for every positive real number a. (ii) The set $\mathbb{P}$ has no largest element, that is, there exists no positive real number β, such that $a \leq \beta$ for every positive real number a.

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- (ii) If β is any positive element, then $\beta < \beta + 1$. This proves the second statement.

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- END OF LECTURE 8.

Lecture 9: Real Numbers : Completeness Axiom

- We are assuming that there is a set called set of real numbers $\mathbb{R}$ with two binary operations', $+$, . , satisfying certain axioms.

Axioms for addition

- A1.

$$a + b = b + a, \quad \forall a, b \in \mathbb{R}.$$

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-Associativity of addition.
- A3. There exists an element called ‘zero’, denoted by ‘0’ in $\mathbb{R}$ such that
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- A4. For every $a \in \mathbb{R}$, there exists an element '$-a$' in $\mathbb{R}$ such that
$$a + (-a) = (-a) + a = 0.$$
-Existence of additive inverse. $-a$ is known as additive inverse of a.

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$$a.a^{-1} = a^{-1}.a = 1.$$
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Distributivity

- D. For a, b, c in $\mathbb{R}$,

$$a.(b + c) = a.b + a.c$$

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- These axioms are known as algebraic axioms. They determine the 'algebraic structure' of real numbers.

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- O3. If $a \in \mathbb{R}$, then exactly one of the following three properties is true:
  (i) $a \in \mathbb{P}$;
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  [This is known as trichotomy property for real numbers.]
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- In particular, $a > 0$ iff $a \in \mathbb{P}$. Similarly $a \geq 0$ iff $a \in \mathbb{P} \bigcup \{0\}$.

Boundedness

- Definition 9.1: A non-empty subset S of $\mathbb{R}$ is said to be bounded above if there exists $u \in \mathbb{R}$ such that
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Examples

- Example 9.4: Consider the set $S = \{1, 2, 3\}$. Then 4 is an upper bound for S. 5 is also an upper bound for S. -1 is a lower bound for S. $\frac{1}{2}$ is also a lower bound for S. Since S admits both lower and upper bounds, it is a bounded subset of $\mathbb{R}$.

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- Example 9.6: It is easily seen that $\mathbb{R}$ is neither bounded below nor bounded above.

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- Proof: Exercise.

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- Remark: Least upper bound, when it exists is unique, for if u_0, u_1 are two least upper bounds, then by $(i), (ii)$ applied to both u_0, u_1, we get $u_0 \leq u_1$ and $u_1 \leq u_0$, and hence $u_0 = u_1$.

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- Example 9.9: Suppose

$$S_1 = \{x \in \mathbb{R} : x \leq 1\};$$

$$S_2 = \{x \in \mathbb{R} : x < 1\}.$$

It is clear that 1 is the least upper bound for both S_1 and S_2. In particular, if u_0 is a least upper bound for S, then u_0 may or may not be in S.

Greatest lower bound

- Definition 9.10: Let S be a non-empty subset of $\mathbb{R}$, which is bounded below. Then $v_0 \in \mathbb{R}$ is said to be a greatest lower bound (or infimum) of S if

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It is clear that 1 is the greatest lower bound for both T_1 and T_2. In particular, if v_0 is a greatest lower bound for S, then v_0 may or may not be in S.

Equivalence

- Proposition 9.12: Let S be a non-empty subset of $\mathbb{R}$. Then the following are equivalent:
  (a) S is bounded above and $u_0 \in \mathbb{R}$ is the least upper bound of S.
  (b) $-S$ is bounded below and $-u_0 \in \mathbb{R}$ is the greatest lower bound of $-S$.

Completeness axiom of $\mathbb{R}$.

- C. Completeness axiom (Least upper bound property): Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

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- Proposition 9.13: Every non-empty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.
- Proof: Suppose $T \subset \mathbb{R}$ is non-empty and is bounded below. Then by consider $-T$ which is bounded above and appeal to the completeness axiom. If u_0 is the least upper bound of $-T$, we know that $-u_0$ is the greatest lower bound of T.

Notation

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$$\sup(S) = \begin{cases} \text{Least upper bound of } S & \text{if S is bounded above;} \\ \infty & \text{otherwise.} \end{cases}$$

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$\sup(S) = -\inf(-S), \quad \inf(S) = -\sup(-S)$
- However, keep in mind that $-\infty, \infty$ are not real numbers.

A Characterization

- Theorem 9.14: Let S be a non-empty subset of $\mathbb{R}$ and let $u_0 \in \mathbb{R}$. Then $u_0 = \sup(S)$ if and only if
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- Proof: Suppose $u_0 = \sup(S)$. Consider any $\epsilon > 0$. Now if every $x \in S$ satisfies $x \leq u_0 - \epsilon$, then $u_0 - \epsilon$ is an upper bound for S. This contradicts the fact that u_0 is the least upper bound. Hence there exists some x_ϵ in S, such that $u_0 - \epsilon < x_\epsilon$.

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- Conversely suppose u_0 satisfies (i) and (ii). Now if u_0 is not the least upper bound of S, then there exists an upper bound u of S such that $u < u_0$. Take $\epsilon = u_0 - u$.
- As u is an upper bound of S, every $x \in S$ satisfies $x \leq u = u_0 - \epsilon$. This violates (ii). So u_0 must be the least upper bound of S.

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- Proof: Suppose $\mathbb{N}$ is bounded above.
- Then by the least upper bound property, $\mathbb{N}$ has a least upper bound, say u_0.

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- As x is a natural number $x + 1$ is also a natural number.
- Then $u_0 < x + 1$ is a contradiction, as u_0 is an upper bound for the set of natural numbers.
- Hence $\mathbb{N}$ can't be bounded above.

A corollary

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- Proof: Let $x \in \mathbb{R}$. If $n \leq x$ for every natural number n, then $\mathbb{N}$ is bounded above by x. Since $\mathbb{N}$ is not bounded above, there exists a natural number n such that $x < n$.

Archimedean property

- Theorem 9.17 (Archimedean property): Suppose $\epsilon \in \mathbb{R}$ and $\epsilon > 0$. Then given any $y \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that

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- That is, $\frac{y}{\epsilon} < n$ or $y < n\epsilon$.

Continuation

- Recall: Archimedean property: Suppose $\epsilon \in \mathbb{R}$ and $\epsilon > 0$. Then given any $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that

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- C- Completeness axiom.

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$$x \leq u, \quad \forall x \in S.$$

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- If S is non-empty and bounded above, its least upper bound is unique and is denoted by $\sup(S)$.

A Lemma

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- Now the result is a special case of Archimedean property with $x = 1$.

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- An integer $n \in \mathbb{Z}$ is said to be an odd number if it is not an even number. Odd integers are all of the form $2k + 1$ for some integer k, and conversely all integers of the form $2k + 1$ with $k \in \mathbb{Z}$ are all odd.

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- Proof. Exercise.

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- Theorem 10.2: There is no rational number x such that $x^2 = 2$.
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- Suppose x is a rational number such that $x^2 = 2$.
- As x is a rational number, $x = \frac{p}{q}$, for some integers, p, q with $q \neq 0$.
- Without loss of generality, we may assume that p, q are relatively prime (they have no common factor bigger than 1). This is possible, because, if $p = rp_1$ and $q = rq_1$, with $r > 1$, we can write $x = \frac{p_1}{q_1}$.

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- Since squares of odd numbers are odd, p also must be even. Say, $p = 2k$, with $k \in \mathbb{Z}$.
- Then we get $4k^2 = 2q^2$ or $2k^2 = q^2$.

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- As $x^2 < 2^2$, we get $x < 2$. Therefore S is bounded above by 2.

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- It is easily seen that $-\sqrt{2}$ is the only other real number whose square 2.

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- $[x]$ is the unique integer satisfying $[x] \leq x < [x] + 1$.
- $x - [x]$ is known as the fractional part of x. Note that

$$0 \leq x - [x] < 1, \quad \forall x \in \mathbb{R}.$$

Intervals

- Notation: For any two real numbers a, b with $a < b$, we write

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- We call (a, b) as open interval and $[a, b]$ as closed interval. Intervals $[a, b)$ etc. are called semi-open intervals.

The density of rational and irrational numbers

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(i) Then there exists a rational number r such that $a < r < b$.
(ii) There exists an irrational number s such that $a < s < b$.
- Proof: (i) Case I: $a = 0$: We know that there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < b$. Since $\frac{1}{n}$ is rational, we are done.

Continuation

- Case II: $a > 0$. Now as $(b - a) > 0$, we can find $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < (b - a)$, or $1 < nb - na$, that is, $na + 1 < nb$.

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- Take $m = [na] + 1$. So $m \in \mathbb{N}$.

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- Case III: $a < 0$. The result for this case can be derived from Case I and Case II (Exercise).

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- END OF LECTURE 10.

Lecture 11: Real Numbers: Nested intervals property and Uncountability

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- This is only a visual aid for us. We are not connecting axioms of geometry with axioms of real line.

Nested Intervals

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- So intersection of a nested family of intervals can be empty.

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- Considering previous examples, the following theorem can be a bit of a surprise.

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- As $I_n \supseteq I_{n+1}$, we have $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ for every n.
- This means that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ for every n.

Continuation

- Since for every n, $I_1 \supseteq I_n$, we get $a_1 \leq a_n \leq b_n \leq b_1$.

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$$a_n \leq u, \qquad (i)$$

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- Combining the last two conclusions, we have

$$a_m \leq b_n, \quad \forall m \qquad (ii)$$

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- Here if $u = v$, then $[u, v]$ is to be understood as the singleton $\{u\}$.

The Singleton

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- Let $\{x_1, x_2, \ldots\}$ be an enumeration of $[a, b]$. (This just means that $n \mapsto x_n$ is a bijective function from $\mathbb{N}$ to $[a, b]$.)

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- As every real number is its integer part plus the fractional part it suffices to consider real numbers in the interval $[0, 1)$ in binary and decimal systems.
- Qn: What is the difference between 1 and $0.9999999\cdots$?
- Ans: $1 = 0.999999\cdots$. In other words, they are equal.

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- This completes the proof by Mathematical Induction.

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- On the other hand if $b_1 = 1$, that is, $x \in [\frac{1}{2}, 1)$, the second binary digit b_2 is 0 if $x \in [\frac{1}{2}, \frac{3}{4})$ and $b_2 = 1$ if $x \in [\frac{3}{4}, 1)$.

Binary expansion: Continuation

- Continuing this way, if $b_1, b_2, \ldots, b_n$ are the first n-binary digits of x, then
$$\frac{b_1}{2} + \frac{b_2}{2^2} \cdots + \frac{b_n}{2^n} \leq x < \frac{b_1}{2^1} + \frac{b_2}{2^2} \cdots + \frac{(b_n + 1)}{2^n}.$$

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- satisfying $x \in I_n$ for every n.

Binary expansion: Continuation

- Continuing this way, if $b_1, b_2, \ldots, b_n$ are the first n-binary digits of x, then

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- In other words, taking

$$I_1 = [\frac{b_1}{2^1}, \quad \frac{(b_1+1)}{2^1}].$$

$$I_2 = [\frac{b_1}{2^1} + \frac{b_2}{2^2}, \quad \frac{b_1}{2^1} + \frac{(b_2+1)}{2^2}]$$

$$I_n = [\frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n}, \quad \frac{b_1}{2^1} + \frac{b_2}{2^2} + \cdots + \frac{(b_n+1)}{2^n}], \quad \forall n,$$

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- In other words, two different real numbers x, y would have different binary expansions.

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- In other words in $(0,1)$, only numbers of the form $\frac{m}{2^k}$, with natural numbers m,k have two binary expansions.
- For instance, $\frac{1}{2}$ is expressed as $0.10000000\ldots$ using the first option and as $0.0111111111\ldots$ through the second option.

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- Similarly $1 = \sup\{\frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^n} : n \in \mathbb{N}\}$.

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- In such cases, we say that x has a terminating decimal expansion. (It ends either with a sequence of 0's or with a sequence of 9's.)

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The sequence $d_1, d_2, \ldots$ is uniquely determined unless $x = \frac{m}{M^k}$ for some natural numbers m, k. Further, if $x = \frac{m}{M^k}$ then x has two possible expressions, one terminating with 0's and another terminating with $(M-1)$'s.

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- END OF LECTURE 12

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- Then by mathematical induction we have a sequence $\{x_1, x_2, \ldots\}$ of distinct elements in S. Clearly $T = \{x_n : n \in \mathbb{N}\}$ is equipotent with $\mathbb{N}$.

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- Conclude that $S \bigcup F$ is equipotent with S.

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- Clearly $T \bigcup C$ is equipotent with T.
- If $f : T \to T \bigcup C$ is a bijection, $\tilde{f} : S \to S \bigcup C$ defined by

$$(\tilde{f})(x) = \begin{cases} f(x) & x \in T; \\ x & x \in S \backslash T \end{cases}$$

is seen to be a bijection from S to $S \bigcup C$ and this completes the proof.

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- By Theorem 13.1, there exists a countably infinite subset T of S.
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- If $f : T \to T \bigcup C$ is a bijection, $\tilde{f} : S \to S \bigcup C$ defined by

$$(\tilde{f})(x) = \begin{cases} f(x) & x \in T; \\ x & x \in S \setminus T \end{cases}$$

  is seen to be a bijection from S to $S \bigcup C$ and this completes the proof.
- Corollary 13.4: If S is an uncountable set and $T \subset S$ is countable then S is equipotent with $S \setminus T$.

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- (v) It is an exercise to cover all the remaining cases.

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- Show that $\mathbb{R} \times \mathbb{R}$ is equipotent with $\mathbb{R}$. More generally, show that $\mathbb{R}^n$ is equipotent with $\mathbb{R}$ for any $n \in \mathbb{N}$.

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- Note that for any element x of X, $f(\{x\}) = \{f(x)\}$, which is the singleton set containing $f(x)$ and is different from the element $f(x)$. This distinction between elements and singleton sets should always be maintained to avoid confusion.

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- Similarly, you can show $f(A) \bigcup f(B) \subseteq f(A \bigcup B)$ and conclude that $f(A \bigcup B) = f(A) \bigcup f(B)$.

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- Proof: (a) follows from the definition of surjectivity. (b) and (c) are interesting exercises.

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Definition and Examples

- Definition 15.1 : A sequence of real numbers

$$a_1, a_2, a_3, \ldots$$

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- Example 15.3 (Fibonacci sequence): This is the sequence:

$$1, 1, 2, 3, 5, 8, \ldots,$$

defined 'recursively', by $a_1 = 1, a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$.

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- Hence $\frac{1}{2}|x - y| = 0$ or $|x - y| = 0$, which is same as saying $x = y$.

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- So the convergence or non-convergence is a property of the whole sequence.

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- We may also write this as: $\lim_{n\to\infty} \frac{1}{n} = 0$.

Boundedness

- Definition 15.7: A sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that

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- Similarly choosing an even number $n \geq K$, we get $|1 - x| < \epsilon$.

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$$|a_n| \leq M, \quad \forall n \in \mathbb{N}.$$
- We have seen that every convergent sequence is bounded but the converse is not true.

Product with a bounded sequence

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- Hence $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.
- Taking $a_n = \frac{1}{n}$ and $b_n = n$, we see that the result may not be true when $\{b_n\}_{n\in\mathbb{N}}$ is not bounded.

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- (d) $\{a_n b_n\}_{n\in\mathbb{N}}$ converges to xy.
- (e) If $b_n \neq 0$ for every $n \in \mathbb{N}$ and $y \neq 0$ then $\{\frac{a_n}{b_n}\}_{n\in\mathbb{N}}$ converges to $\frac{x}{y}$.

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- Clearly (c) follows from (a) and (b).

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- If $x \neq 0$, this can be done by taking $\epsilon' = \frac{\epsilon}{2|x|}$, and using convergence of $\{b_n\}$. If $x = 0$, the inequality is trivially true and we can simply take $K_2 = 1$.

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$$|b_n - y| < \frac{|y|}{2}, \quad \forall n \geq K.$$

- This implies that $|b_n| \geq \frac{|y|}{2}$ for $n \geq K$. (Why?)
- Therefore $\frac{1}{|b_n|} \leq \frac{2}{|y|}$ for $n \geq K$.
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$$M = \max\{\frac{1}{|b_1|}, \frac{1}{|b_2|}, \ldots, \frac{1}{|b_{K-1}|}, \frac{2}{|y|}\}.$$

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- Claim: There exists $M > 0$ such that $\frac{1}{|b_n|} \leq M$ for all $n \in \mathbb{N}$.
- Proof of claim: Recall that $\lim_{n\to\infty} b_n = y$ and $y \neq 0$.
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- Note that M is well-defined as $b_n \neq 0$ for every n.
- Now we have $\frac{1}{|b_n|} \leq M$ for every $n \in \mathbb{N}$.

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$$|a_n - x| < \epsilon, \quad \forall n \geq K.$$

  In such a case, $\{a_n\}_{n\in\mathbb{N}}$ is said to converge to x, and x is said to be the limit of $\{a_n\}_{n\in\mathbb{N}}$.
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- We have seen that every convergent sequence is bounded but the converse is not true.

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- (c) For $c, d \in \mathbb{R}$, $\{ca_n + db_n\}_{n\in\mathbb{N}}$ converges to $cx + dy$.
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- (d) $\{a_n b_n\}_{n\in\mathbb{N}}$ converges to xy.
- (e) If $b_n \neq 0$ for every $n \in \mathbb{N}$ and $y \neq 0$ then $\{\frac{a_n}{b_n}\}_{n\in\mathbb{N}}$ converges to $\frac{x}{y}$.

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- Theorem 17.1: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence converging to x and $a_n \geq 0$ for every $n \in \mathbb{N}$. Then $x \geq 0$.

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- So we have a contradiction. Hence $x < 0$ is not possible.

Order

- Theorem 17.2: Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences converging to x, y respectively. Suppose $a_n \leq b_n$ for every n. Then $x \leq y$.

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- Also $c_n \geq 0, \forall n$.
- Hence by previous theorem $y - x \geq 0$, or equivalently $x \leq y$.
- Warning: In this Theorem, $a_n < b_n$ for all n does not imply $x < y$. For example, take $a_n = 0$ and $b_n = \frac{1}{n}$ for all n. Then $x = y = 0$ and we don't have $x < y$.

Squeeze theorem

- Theorem 17.3 (Squeeze theorem): Suppose $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ are three sequences satisfying $a_n \leq b_n \leq c_n, \quad \forall n \in \mathbb{N}$.

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- Take $K = \max\{K_1, K_2\}$.
- Now for $n \geq K$, as $a_n \leq b_n \leq c_n$, we get
$$x-\epsilon < a_n \leq b_n \leq c_n < x+\epsilon.$$
- In particular, $b_n \in (x-\epsilon, x+\epsilon), \quad \forall n \geq K$ or $|b_n - x| < \epsilon, \quad \forall n \geq K$.
- Hence $\{b_n\}_{n\in\mathbb{N}}$ converges to x.

Monotonicity

- Definition 17.4: A sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers is said to be increasing (or non-decreasing) if

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- Example 17.5: The sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a decreasing sequence. The sequence $\{n\}_{n\in\mathbb{N}}$ is an increasing sequence.

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- Example 17.5: The sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a decreasing sequence. The sequence $\{n\}_{n\in\mathbb{N}}$ is an increasing sequence.
- Note that an increasing sequence is always bounded below by the first term, that is, $a_1 \leq a_n, \quad \forall n \in \mathbb{N}$ and similarly a decreasing sequence is always bounded above by the first term.

Bounded monotonic sequences

- Theorem 17.6: (i) An increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ is convergent if and only if it is bounded above. In such a case,

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

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- Proof: Clearly (iii) follows from (i) and (ii).

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- (iii) A monotonic sequence is convergent if and only if it is bounded.
- Proof: Clearly (iii) follows from (i) and (ii).
- Also (ii) follows from (i), by considering $\{-a_n\}_{n\in\mathbb{N}}$. So it suffices to prove (i).

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- Take $x = \sup\{a_n : n \in \mathbb{N}\}$.
- We want to show that $\{a_n\}_{n\in\mathbb{N}}$ converges to x.
- Take any $\epsilon > 0$. Then $x - \epsilon < x$.

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- Take $x = \sup\{a_n : n \in \mathbb{N}\}$.
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- Now the result $y = \lim_{n\to\infty} a_n$, is clear from the previous theorem.

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- Inductively, one can show that

$$a = a_1 \leq a_2 \leq \cdots a_n \leq b_n \leq \cdots \leq b_2 \leq b_1 = b.$$

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- This value is known as arithmetic-geometric mean of a and b. 7.

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- END OF LECTURE 17.

Lecture 18. Bolzano-Weierstrass theorem

- We recall a few notions from the previous lecture.
- Definition 17.4: A sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers is said to be increasing (or non-decreasing) if

$$a_n \le a_{n+1}, \quad \forall n \in \mathbb{N}.$$

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- Example 17.5: The sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a decreasing sequence. The sequence $\{n\}_{n\in\mathbb{N}}$ is an increasing sequence.
- Note that an increasing sequence is always bounded below by the first term, that is, $a_1 \leq a_n, \quad \forall n \in \mathbb{N}$ and similarly a decreasing sequence is always bounded above by the first term.

Bounded monotonic sequences

- Theorem 17.6: (i) An increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ is convergent if and only if it is bounded above. In such a case,

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- (iii) A monotonic sequence is convergent if and only if it is bounded.

Subsequences

- Definition 18.1: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Let

$$n_1 < n_2 < n_3 < \cdots$$

be a strictly increasing sequence of natural numbers. Then $\{a_{n_k}\}_{k\in\mathbb{N}}$ or equivalently,

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- Such subsequences are known as tails of the given sequence.

Subsequences of convergent sequences

- Theorem 18.4: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers converging to some $x \in \mathbb{R}$. Then every subsequence of $\{a_n\}_{n\in\mathbb{N}}$ converges to x. In particular, every tail of this sequence converges to x.

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- Hence $\{a_{n_k}\}_{k\in\mathbb{N}}$ converges to x.

Limit points

- Definition 18.5: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then $y \in \mathbb{R}$ is said to be limit point of $\{a_n\}_{n\in\mathbb{N}}$, if it has a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ converging to y.

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- Note that the previous Theorem tells us that if $\{a_n\}_{n\in\mathbb{N}}$ is a sequence converging to x, then its set of limit points is the singleton $\{x\}$.

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- It is an exercise to show that the set of limit points of this sequence is given by

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- In other words, we have an increasing subsequence in:
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Sequential Compactness

- Theorem 18.11: Suppose $[a, b]$ is an interval and $\{c_n\}_{n \in \mathbb{N}}$ is a sequence of real numbers with $c_n \in [a, b]$. Then $\{c_n\}_{n \in \mathbb{N}}$ has a convergent subsequence and any such subsequence converges to a point in $[a, b]$.

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- Note that the same property does not hold for intervals like (a, b) as the limit may not be an element of the interval.

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- We may write $|a_m - a_n| < \epsilon$ equivalently as $a_m \in (a_n - \epsilon, a_n + \epsilon)$ or as $(a_m - a_n) \in (-\epsilon, +\epsilon)$.

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- In particular, $|a_m| < |a_K| + 1, \quad \forall m \geq K.$

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- Take $\epsilon = 1$. Using Cauchy property, choose $K \in \mathbb{N}$ such that

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- Using convergence of $\{a_{n_k}\}_{k\in\mathbb{N}}$, choose K_2 such that

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- This completes the proof.

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- There is a way of completing every metric space and if we complete $\mathbb{Q}$ by this procedure we get the set of real numbers $\mathbb{R}$. This is one way of constructing $\mathbb{R}$.

Infinite series

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  is meaningful.
- Definition 19.5: Suppose $a_1, a_2, \ldots$ are real numbers. Take $s_n = \sum_{j=1}^{n} a_j$. Here $\{s_n\}_{n\in\mathbb{N}}$ are known as partial sums of the series. If $\lim_{n\to\infty} s_n$ exists then the series, $\sum_{j=1}^{\infty} a_j$ is said to converge and

$$\sum_{j=1}^{\infty} a_j := \lim_{n\to\infty} s_n.$$

  If $\lim_{n\to\infty} s_n$ does not exist, the series $\sum_{j=1}^{\infty} a_j$ is said to diverge.

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- Now

$$\begin{aligned} s_n &:= \sum_{j=1}^{n} \frac{1}{2^j} \\ &= \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \\ &= \frac{1}{2}[1 + \frac{1}{2} + \cdots + (\frac{1}{2})^{(n-1)}] \\ &= \frac{1}{2} . \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} \\ &= 1 - \frac{1}{2^n} \end{aligned}$$

Continuation

- Using Bernoulli's inequality, we have seen that $\frac{1}{2^n} < \frac{1}{n+1}$ and hence $\lim_{n\to\infty} \frac{1}{2^n} = 0$. Hence $\lim_{n\to\infty} s_n = 1$.

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- Similarly, one can show that for any $|r| < 1$, $\lim_{n\to\infty} r^{n-1} = 0$ and
$$1 + r + r^2 + \cdots = \frac{1}{1-r}$$

Convergence

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- The converse is not true is seen by considering the 'Harmonic series' :
- $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges as the corresponding partial sums are unbounded.

Alternating sum

- Theorem 19.8: A series $\sum_{j=1}^{\infty} a_j$, where $a_j = (-1)^{j+1} b_j$, with a decreasing sequence $\{b_j\}_{j \in \mathbb{N}}$ of positive real numbers is convergent if and only if $\lim_{n\to\infty} b_n = 0$.

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- We have, $s_{2k+2} = s_{2k} + b_{2k+1} - b_{2k+2}$.

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- That is,

$$b_1 - b_2 = s_2 \leq s_4 \leq \cdots \leq s_{2k} \leq s_{2k-1} \leq \cdots s_3 \leq s_1 = b_1$$

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- So $\lim_{k\to\infty} s_{2k}$ and $\lim_{k\to\infty} s_{2k-1}$ exist.

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- As every bounded monotonic sequence is convergent, this subsequence is convergent. This completes the proof.

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- Definition 18.5: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then $y \in \mathbb{R}$ is said to be limit point of $\{a_n\}_{n\in\mathbb{N}}$, if it has a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ converging to y.
- We would like to understand the structure of limit points better. The following theorem is easy to prove.

Terms around a limit point

- Theorem 20.1: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then $y \in \mathbb{R}$ is a limit point of the sequence $\{a_n\}_{n\in\mathbb{N}}$ if and only if the set

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- The converse is also easy to see.

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- In conclusion, $\{b_n\}$ is a bounded decreasing sequence. Hence $\lim_{n\to\infty} b_n$ exists.

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- Definition 20.2: For any bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, the $\lim_{n\to\infty} b_n$ defined as above is known as the limit superior or limsup of the bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, and we write:

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- A bounded sequence may not be convergent and so it may not have a limit. But it always has liminf and limsup.

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- Example 20.4: Consider the sequence $\{a_n\}$ where,

$$a_n = \begin{cases} 5 & \text{if } n = 3k+1, k \in \mathbb{N} \bigcup \{0\} \\ 6 & \text{if } n = 3k+2, k \in \mathbb{N} \bigcup \{0\} \\ 7 & \text{if } n = 3k, k \in \mathbb{N}. \end{cases}$$

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- In particular, it is not immediate that limsup and liminf are limit points of the sequence.

A Characterization

- Theorem 20.6: Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers and suppose $z = \limsup_{n\to\infty} a_n$. Then for every $\epsilon > 0$, the set

$$S_+(z,\epsilon) = \{n : a_n > z+\epsilon\} \text{ is finite.} \quad (*)$$

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- Fix $\epsilon > 0$. Take $b_n = \sup\{a_m : m \geq n\}$. By the definition of limsup, $z = \lim_{n\to\infty} b_n$.

A Characterization

- Theorem 20.6: Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers and suppose $z = \limsup_{n\to\infty} a_n$. Then for every $\epsilon > 0$, the set

$$S_+(z,\epsilon) = \{n : a_n > z+\epsilon\} \text{ is finite.} \quad (*)$$

and the set

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- Hence there exists $K \in \mathbb{N}$ such that

$$b_n \in (z-\epsilon, z+\epsilon), \quad \forall n \geq K.$$

Continuation

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- Therefore $b_n \leq v + \epsilon$ for $n \geq M_\epsilon$. Hence $z = \lim_{n\to\infty} b_n \leq v + \epsilon$.

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- This allows us to choose a subsequence $\{a_{n_r}\}_{r\in\mathbb{N}}$, where $v - \frac{1}{r} < a_{n_r}$. Then $v - \frac{1}{r} < b_{n_r}$, and hence on taking limit as $r \to \infty$, $v \leq \lim_{r\to\infty} b_{n_r} = z$. That is, $v \leq z$. Combining the two statements we have $v = z$.

limsup as a limit point

- Theorem 20.7: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence of real numbers. Then $\limsup_{n\to\infty} a_n$ is a limit point of $\{a_n\}_{n\in\mathbb{N}}$ and if y is any limit point of $\{a_n\}_{n\in\mathbb{N}}$, then $y \leq \limsup_{n\to\infty} a_n$.

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- In conclusion, $\{b_n\}$ is a bounded decreasing sequence. Hence $\lim_{n\to\infty} b_n$ exists.

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- A bounded sequence may not be convergent and so it may not have a limit. But it always has liminf and limsup.

A Characterization

- Theorem 20.6: Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers and suppose $z = \limsup_{n\to\infty} a_n$. Then for every $\epsilon > 0$, the set

$$S_+(z,\epsilon) = \{n : a_n > z + \epsilon\} \text{ is finite.} \quad (*)$$

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Limit superior as a limit point

- Theorem 20.7: Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence of real numbers. Then $\limsup_{n\to\infty} a_n$ is a limit point of $\{a_n\}_{n\in\mathbb{N}}$ and if y is any limit point of $\{a_n\}_{n\in\mathbb{N}}$, then $y \leq \limsup_{n\to\infty} a_n$.

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- Hence z is a limit point of $\{a_n\}_{n\in\mathbb{N}}$.
- The fact that z is the largest limit point is also clear from the characterization for if $z < v$, then taking $\epsilon = \frac{v-z}{2}$, $(v-\epsilon, v+\epsilon) \subseteq S_+(z,\epsilon)$ has finitely many terms of the sequence.

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- Results similar to that of limsup hold for liminf. These can be proved by similar methods or by observing that

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the result follows by the squeeze theorem.
- This shows that when we do not know whether a sequence is convergent or not, we may try to compute its liminf and limsup and see whether they are equal or not.

Properly divergent sequences

- Definition 21.3: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Then it is said to properly diverge to $+\infty$ if for every $M \in \mathbb{R}$ there exists $K \in \mathbb{N}$ such that
$$a_n \geq M, \quad \forall n \geq K.$$
This is written as:
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- Example 21.4: Define:

$$a_n = n^2, \quad \forall n \in \mathbb{N}.$$

$$b_n = \begin{cases} 5 & \text{if } n \text{ is odd.} \\ n & \text{if } n \text{ is even.} \end{cases}$$

$$c_n = \begin{cases} 5 & \text{if } n \text{ is odd.} \\ 6 & \text{if } n \text{ is even.} \end{cases}$$

Here $\{a_n\}_{n\in\mathbb{N}}$ is properly divergent to $+\infty$, $\{b_n\}_{n\in\mathbb{N}}$ is unbounded and divergent but it is not properly divergent, $\{c_n\}_{n\in\mathbb{N}}$ is bounded and divergent but not properly divergent.

Basic properties

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- Also for $n \geq K$, $a_n b_n \geq M.1 = M$. Therefore, $\{a_n b_n\}$ properly diverges to $+\infty$.

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- Proofs of other claims are left out as exercises.

Some more properties

- Theorem 21.6: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequences of real numbers properly diverging to $+\infty$ and let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence converging to some real number x.

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- (iii) If $x > 0$ and $b_n \neq 0$ for every n, then $\{\frac{a_n}{b_n}\}$ properly diverges to ∞. If $x < 0$ and $b_n \neq 0$ for every n, then $\{\frac{a_n}{b_n}\}$ properly diverges to $-\infty$.

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- (iv) If $a_n \neq 0$ for every n, then $\{\frac{b_n}{a_n}\}$ converges to 0.

Continuation

- If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ properly diverge to $+\infty$, $\{a_n - b_n\}_{n\to\infty}$ may not converge. Similarly $\{\frac{a_n}{b_n}\}_{n\in\mathbb{N}}$ need not converge.

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- Give examples to illustrate such phenomenon.

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- END OF LECTURE 21

Lecture 22. Continuous functions

- Definition 22.1: Let $A \subseteq \mathbb{R}$ and let $c \in A$. Then a function $f : A \to \mathbb{R}$ is said to be continuous at c, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

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- Example 22.3: Define $f : [0,1] \to \mathbb{R}$ by

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- This shows that $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(c)$.

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- This completes the proof

More Examples

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$$g(x) = \begin{cases} 0 & \text{if } \; x = 1; \\ 7 & \text{if } \; x \in [2,3]. \end{cases}$$

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- Remark 22.6: Suppose $A \subset \mathbb{R}$ and $c \in A$ is isolated in A. Then every function $f : A \to \mathbb{R}$ is continuous at c.

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- Proof. (i) For $\epsilon > 0$, using continuity of f at c, choose $\delta_1 > 0$, such that

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- Now take $\delta = \min\{\delta_1, \delta_2\}$. Then for $x \in (c - \delta, c + \delta) \bigcap A$, we get

$$|f(x)+g(x)-f(c)-g(c)| \leq |f(x)-f(c)|+|g(x)-g(c)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

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- Proof. (i) For $\epsilon > 0$, using continuity of f at c, choose $\delta_1 > 0$, such that

$$|f(x) - f(c)| < \frac{\epsilon}{2}, \quad \forall x \in (c - \delta_1, c + \delta_1) \bigcap A.$$

- Similarly using continuity of g at c, choose $\delta_2 > 0$ such that

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- Now take $\delta = \min\{\delta_1, \delta_2\}$. Then for $x \in (c - \delta, c + \delta) \bigcap A$, we get

$$|f(x)+g(x)-f(c)-g(c)| \leq |f(x)-f(c)|+|g(x)-g(c)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

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$$|f(x)+g(x)-f(c)-g(c)| \leq |f(x)-f(c)|+|g(x)-g(c)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$
- Therefore $f + g$ is continuous at c.
- It is easy to see that if f is continuous at c, af is continuous at c. Similarly bg is continuous at c. Combining with the previous result, $af + bg$ is continuous at c.

Alternative Proof

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- Hence, $\{af(x_n) + bg(x_n)\}_{n\in\mathbb{N}}$ converges to $af(c) + bg(c)$.

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- Similarly, $\{f(x_n)g(x_n)\}$ converges to $f(c)g(c)$ and if $g(x) \neq 0$ for every x, $\{\frac{f(x_n)}{g(x_n)}\}$ converges to $\frac{f(c)}{g(c)}$.

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- Hence fg and $\frac{f}{g}$ are continuous. This completes the proof.

Algebra of continuous functions

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$$(af + bg)(x) = af(x) + bg(x), \quad \forall x \in A,$$

is continuous.

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is continuous.
- (iii) If $g(x) \neq 0, \ \ \forall x \in A$, then $\frac{f}{g}$ defined by
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \ \ \forall x \in A,$$
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$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \ \ \forall x \in A,$$
is continuous.
- Proof: This is clear from the previous theorem and the definition of continuous functions.

Restrictions of continuous functions

- Theorem 23.3: Let $A \subseteq \mathbb{R}$ and let B be a subset of A and let $c \in B$. Suppose $f : A \to \mathbb{R}$ is a function continuous at c. Then $g : B \to \mathbb{R}$ defined by

  $$g(x) = f(x), \quad \forall x \in B,$$

  is continuous at c. If f is continuous, then g is continuous.

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- Proof: This is obvious from the definition of continuity.
- Notation: The function g of this theorem is called the restriction of f to B and is denoted by $f|_B$.

Continuity of polynomials

- Theorem 23.4: Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial defined by

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad \forall x \in \mathbb{R},$$

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- where $n \in \mathbb{N}\bigcup\{0\}$ and $a_0, a_1, \ldots, a_n$ are real numbers. Then p is continuous.
- Proof: It is easy to see that the constant function

$$p_0(x) = a_0, \quad x \in \mathbb{R}$$

and the identity function,

$$p_1(x) = x, \quad x \in \mathbb{R}$$

are continuous. Now by (ii) of Theorem 23.2, and mathematical induction, the polynomials

$$p_k(x) = x^k, \quad \forall x \in \mathbb{R}$$

$k \in \mathbb{N}$, are continuous. The proof is complete by a simple application of (i) of Theorem 23.2.

Rational functions

- Corollary 23.5: For any non-empty subset B of $\mathbb{R}$ and any real polynomial p, $p|_B$, defined by

$$p|_B(x) = p(x), \quad x \in B$$

is continuous.

Rational functions

- Corollary 23.5: For any non-empty subset B of $\mathbb{R}$ and any real polynomial p, $p|_B$, defined by

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- If q is another polynomial such that $q(x) \neq 0$ for $x \in B$, then $\frac{p|_B}{q|_B}$ is a continuous function on B.

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- Example 23.6: The function $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}, \quad \forall x \in \mathbb{R}\backslash\{0\}$ is continuous.

Composition of continuous functions

- Theorem 23.7: Let A, B be subsets of $\mathbb{R}$ and $c \in A$. Suppose f, g are real valued functions on A, B respectively and $f(A) \subseteq B$. Suppose f is continuous at c and g is continuous at $f(c)$. Then $h = g \circ f$ is continuous at c.

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- In other words $\{h(x_n)\}$ converges to $h(c)$. This proves that h is continuous at c.
- Exercise 23.8: Prove the previous theorem directly using the definition of continuity.

Composition of continuous functions

- Theorem 23.9: Let A, B be subsets of $\mathbb{R}$. Suppose f, g are continuous real valued functions on A, B respectively and $f(A) \subseteq B$. Then $h = g \circ f$ is a continuous function.

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- Theorem 23.9: Let A, B be subsets of $\mathbb{R}$. Suppose f, g are continuous real valued functions on A, B respectively and $f(A) \subseteq B$. Then $h = g \circ f$ is a continuous function.
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- Example 23.10 (Dirichlet function): Define $d : \mathbb{R} \to \mathbb{R}$ by

$$d(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

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- Then d is discontinuous at every $x \in \mathbb{R}$.
- Example 23.11: Define $g : [1, 2] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational;} \\ \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \quad p, q \in \mathbb{N} \\ & \quad p, q \;\; \text{relatively prime.} \end{cases}$$

Then g is continuous at irrational points in $[1, 2]$, but is discontinuous at rational points in $[1, 2]$.

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- END OF LECTURE 23.

Lecture 24. Continuous functions on intervals

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- Definition 24.1: Let A be a non-empty set and let $f : A \to \mathbb{R}$ be a function. Then f is said to be bounded if

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In such a case M said to be a bound for f.

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Examples

- Example 24.2: Let $f : [0, 1) \to \mathbb{R}$ be the function $f(x) = x, \quad \forall x \in [0, 1)$. Then f is bounded with bound 1. $\sup(f)$ is not a maximum. However, inf is a minimum with $\inf(f) = f(0)$.

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- Example 24.3: Let $g : (0, 1) \to \mathbb{R}$ be the function $g(x) = \frac{1}{x}, \quad x \in (0, 1)$. Then f is continuous but not bounded.

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- Now $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in $[a, b]$.

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- Proof: Suppose $f : [a, b] \to \mathbb{R}$ is not bounded. We want to arrive at a contradiction.
- As f is not bounded, for every $n \in \mathbb{N}$ there exists some x_n in $[a, b]$ such that $|f(x_n)| \geq n$.
- Now $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in $[a, b]$.
- Then by Bolzano-Weierstrass theorem there exists a convergent subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$.

Continuation

- Suppose $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges to $c \in [a, b]$.

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- We have already seen that continuous functions on open intervals need not be bounded. Also examples, such as $f(x) = x$, show that continuous functions on $\mathbb{R}$ need not be bounded.

Maximum and minimum

- Theorem 24.5: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists c, d in $[a, b]$ such that

$$f(c) = \sup\{f(x) : x \in [a, b]\};$$

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- By squeeze theorem,

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- Continuing this way, after choosing $I_n = [a_n, b_n]$, either $f(\frac{a_n+b_n}{2}) = 0$ or we have $I_{n+1} = [a_{n+1}, b_{n+1}]$, in such a way that $I_n \supset I_{n+1}$ with $(b_{n+1} - a_{n+1}) = \frac{1}{2}(b_n - a_n)$.

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- Suppose $\{c\} = \bigcap_{n\in\mathbb{N}} I_n$.
- We clearly have $\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$.

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- In this proof we have seen a way of locating the root by successively bisecting the interval.

Intermediate value theorem

- Theorem 25.2: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) < z < f(b)$ or $f(a) > z > f(b)$, then there exists $c \in (a, b)$ such that $f(c) = z$.

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- If $f(a) > z > f(b)$, consider g defined by

$$g(x) = z - f(x), \quad x \in [a, b]$$

and similar proof works.

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- Clearly, p is continuous and is unbounded.
- Therefore, we can get a b such that $t < p(b)$. (Exercise: We may take $b = t + 1$.)

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- For $0 < c < d$,

$$\begin{aligned} d^n - c^n &= (d-c)(d^{n-1} + cd^{n-2} + c^2 d^{n-s} + \cdots + c^{n-1}) \\ &= (d-c)(\sum_{j=0}^{n-1} c^j d^{n-1-j})) > 0. \end{aligned}$$

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- In other words if $0 < c < d$, we have $c^n < d^n$ and so we can't have $c^n = d^n$.. This shows the uniqueness of positive n^{th} root of t.

Roots of polynomials

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- Exercise 25.5: Suppose p is an odd degree real polynomial. Show that there exists a real number λ such that $p(\lambda) = 0$.

Continuous image of an interval

- Theorem 25.6: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then
$$f([a, b]) = [s, t]$$
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- But this is clear from the inter mediate value theorem as there exist c, d in $[a, b]$ such that $f(c) = s$ and $f(d) = t$.

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- Theorem 25.7: Suppose $I \subseteq \mathbb{R}$ is an interval, and $f : I \to \mathbb{R}$ is a continuous function. Then $f(I)$ is an interval.
- Recall that intervals are sets of the form $\{a\}, [a, b], [a, b), (a, b], [a, \infty), (a, \infty), (-\infty, b], (-\infty, b), (-\infty, \infty)$, with $a, b \in \mathbb{R}, a < b$.

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- Exercise 25.8: Show that a non-empty subset S of $\mathbb{R}$ is an interval if and only if $x, y \in S$ with $x < y$ implies $[x, y] \subseteq S$.
- Now the proof of Theorem 25.7 follows easily from the inter mediate value theorem.

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- Sketch of proof:
- We model the equator by a circle, or by the interval $[0, 1]$, where we identify the points 0 and 1.

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- Suppose $f : A \to \mathbb{R}$ is continuous at every y in A. Then we have for every $\epsilon > 0$, there exists δ, depending on y, such that
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- It is important here that the δ here depends only on ϵ and not on x or y.

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- Exercise 26.4: Show that $f : (0,1) \to (0,1)$ defined by
$$f(x) = \frac{1}{x}, \;\; \forall x \in (0,1),$$
is not uniformly continuous.

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  holds.
- In particular, this inequality does not hold for $\delta = \frac{1}{n}$ for every $n \in \mathbb{N}$.
- This means that there exist x_n, y_n in $[a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ and

$$|f(x_n) - f(y_n)| \geq \epsilon_0.$$

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- Take $K = \max\{K_1, K_2\}$. Then by triangle inequality we have,

$$|f(z_K) - f(w_K)| \leq |f(z_K) - f(z)| + |f(z) - f(w_K)| < \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0$$

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Continuous bijections

- Theorem 26.7: Let a, b, a', b' be real numbers with $a < b$ and $a' < b'$. If $f : [a, b] \to [a', b']$ is a continuous bijection then either f is strictly increasing with $f(a) = a'$ and $f(b) = b'$ or f is strictly decreasing with $f(a) = b'$ and $f(b) = a'$

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- Proof: We know that any continuous function f on $[a, b]$ maps $[a, b]$ onto $[s, t]$ where
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- We claim that if $c < d$, then f is strictly increasing. By intermediate value theorem, $f([c, d]) = [a', b']$. Now the bijectivity of f forces $c = a$ and $d = b$, so that $f(a) = a'$ and $f(b) = b'$.

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- If f is not strictly increasing, there exist x, y in $[a, b]$ such that $x < y$ and $f(x) > f(y)$ (Since f is injective $f(x) = f(y)$ is ruled out.)

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Continuation

- If f is not strictly increasing, there exist x, y in $[a, b]$ such that $x < y$ and $f(x) > f(y)$ (Since f is injective $f(x) = f(y)$ is ruled out.)
- Since $f(a) = a'$ and $f(x) > f(y)$, $x = a$ is not possible.
- So we have $a < x < y \leq b$ and $f(a) = a'$, and $f(x) > f(y) > a'$
- On applying intermediate value theorem to $f|_{[a,x]}$ there must be some $z \in [a, x]$ such that $f(z) = f(y)$. This contradicts injectivity of f.
- Therefore if $c < d$, then f is strictly increasing and $f(a) = a', f(b) = b'$.
- Similarly if $d < c$, f is strictly decreasing and $f(a) = b', f(b) = a'$.
- Finally $c = d$ is not possible as f can't be a constant function due to injectivity of f. ■
- END OF LECTURE 26.

Lecture 27. Limits to cluster points

- Definition 27.1: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is said to be a cluster point (or accumulation point) of A if for every $\delta > 0$
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- Proposition 27.3: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is a cluster point of A if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A\backslash\{c\}$ converging to c.

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- Note that we are excluding c from these sequences.

Limits of functions to cluster points

- Definition 27.4: Let c be a cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then f is said to have a limit at c if there exists $z \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

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- Notation: If z is the limit of f at c, we write

$$\lim_{x \to c} f(x) = z.$$

Sequential version

- Proposition 27.5: Let c be a cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then z is limit of f at c if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A\backslash\{c\}$ converging to c, $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to z.

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- Then for $n \geq K$, $x_n \in (c-\delta, c+\delta)\bigcap(A\backslash\{c\})$ and hence $|f(x_n) - z| < \epsilon, \quad \forall n \geq K.$

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- Then for $n \geq K$, $x_n \in (c-\delta, c+\delta)\bigcap(A\backslash\{c\})$ and hence $|f(x_n) - z| < \epsilon, \quad \forall n \geq K$.
- Therefore $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(c)$.

Continuation

- Now suppose z is not a limit of f at c. Then there exists $\epsilon_0 > 0$ such that for no $\delta > 0$

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- Clearly then $\{x_n\}_{n \in \mathbb{N}}$ converges to c, but $\{f(x_n)\}$ does not converge to z. $\blacksquare$.

Example

- Example 27.6: Define $h : [0,2)\bigcup(2,3] \to \mathbb{R}$ by

$$h(x) = \begin{cases} 2x & \text{if } x \in [0,2) \\ \frac{(x^3-2x^2)}{x-2} & \text{if } x \in (2,3] \end{cases}$$

extends to a continuous function $\tilde{h}$ on $[0,3]$ by taking $\tilde{h}(x) = h(x)$ for $x \in [0,2)\bigcup(2,3]$ and $\tilde{h}(2) = 4$.

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- Remark: Suppose c is a cluster point of a set $A \subseteq \mathbb{R}$ and $f; A \to \mathbb{R}$ is a function. Suppose $\lim_{x\to c} f(x) = z$, then $\tilde{f} : A\bigcup\{c\} \to \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A\backslash\{c\} \\ z & \text{if } x = c \end{cases}$$

  is continuous at c.

Left and right hand cluster points

- Definition 27.7: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is said to be a right cluster point of A if for every $\delta > 0$

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Similarly c is said to be a left cluster point of A if for every $\delta > 0$

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- Proof. Exercise.

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- Definition 27.9: Let c be a right cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then f is said to have a right hand limit at c if there exists $z \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

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Monotonic functions

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- (iii) For every $c \in (a, b)$

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Therefore f is continuous at c if and only if

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- Consider any $\epsilon > 0$. Since $z - \epsilon$ is less than the supremum there exists $d \in [a, c)$ such that

$$z - \epsilon < f(d) \leq z.$$

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$$z = \sup\{f(x) : x \in [a, c)\}.$$
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- END OF LECTURE 27.

Lecture 28. Inverses of continuous bijections and extensions of functions

- Definition 27.1: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is said to be a cluster point (or accumulation point) of A if for every $\delta > 0$

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- Proposition 27.3: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is a cluster point of A if and only if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \backslash \{c\}$ converging to c.

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- Note that we are excluding c from these sequences.

Limits of functions at cluster points

- Definition 27.4: Let c be a cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then f is said to have a limit at c if there exists $z \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - z| < \epsilon, \quad \forall x \in (c - \delta, c + \delta) \bigcap (A \backslash \{c\}).$$

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- Proposition 27.5: Let c be a cluster point of a subset A of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Then z is limit of f at c if and only if for every sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \backslash \{c\}$ converging to c, $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to z.

Left and right hand cluster points

- Definition 27.7: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is said to be a right cluster point of A if for every $\delta > 0$

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- Proposition 27.8: Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then the following are equivalent:
- (i) c is a right cluster point of A.
- (ii) There exists a sequence $\{x_n\}$ in $A \bigcap (c, \infty)$ converging to c.

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- (i) c is a right cluster point of A.
- (ii) There exists a sequence $\{x_n\}$ in $A \bigcap (c, \infty)$ converging to c.
- (iii) There exists a strictly decreasing sequence $\{x_n\}$ in A converging to c.

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- Theorem 27.11: Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a function. Suppose f is increasing then the following hold.

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- (iii) For every $c \in (a, b)$

$$\lim_{x \to c-} f(x) \leq f(c) \leq \lim_{x \to c+} f(x).$$

Therefore f is continuous at c if and only if

$$\lim_{x \to c-} f(x) = \lim_{x \to c+} f(x).$$

Inverses of monotone continuous functions

- Theorem 28.1: Let a, b, a', b' be real numbers with $a < b$ and $a' < b'$. Let $f : [a, b] \to [a', b']$ be a continuous bijection with $f(a) = a'$ and $f(b) = b'$. Then $f^{-1} : [a', b'] \to [a, b]$ is a continuous bijection.

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- Also $f^{-1}(a') = a$ and $f^{-1}(b') = b$.
- Further, we know that f is strictly increasing.
- This implies, that f^{-1} is also strictly increasing as for $y < y'$ if $f^{-1}(y) \geq f^{-1}(y')$, on applying f we get $y \geq y'$, contradicting $y < y'$.

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- Then for any $c' \in (a', b']$

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- Similarly, for every $c' \in [a', b')$, $\lim_{y \to c+} f^{-1}(y) = f^{-1}(c')$.

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- Therefore $f|_{[a,c]} : [a, c] \to [a', c']$ is a bijection.
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- Hence, $\lim_{y \to c'-} f^{-1}(y) = f^{-1}(c)$.
- Similarly, for every $c' \in [a', b')$, $\lim_{y \to c+} f^{-1}(y) = f^{-1}(c')$.
- Therefore f^{-1} is continuous.

n^{th}-root function

- Example 28.2: For any $n \in \mathbb{N}$, and any $T > 0$, the function $p : [0, T] \to [0, T^n]$ defined by $p(x) = x^n$ is a continuous bijection.

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- Hence $q = p^{-1} : [0, T^n] \to [0, T]$ defined by $q(y) = y^{\frac{1}{n}}$ is a continuous bijection.
- It follows that $q : [0, \infty) \to [0, \infty)$ defined by $q(x) = x^{\frac{1}{n}}$ is a continuous bijection.

Extensions of uniformly continuous functions

- Theorem 28.3: Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : (a, b) \to \mathbb{R}$ be a function. Then there exists unique continuous function $\tilde{f} : [a, b] \to \mathbb{R}$ such that $\tilde{f}(x) = f(x), \quad \forall x \in (a, b)$ if and only if f is uniformly continuous.

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- To prove the converse we need a lemma which is of independent interest.

Cauchy property

- Lemma 28.4: Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be uniformly continuous. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in A. Then $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

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- Proof. Consider $\epsilon > 0$.

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- Proof. Consider $\epsilon > 0$.
- Then as f is uniformly continuous, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon, \quad \forall x, y \in A, \quad \text{with} \quad |x - y| < \delta$$

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- In other words, uniformly continuous functions map Cauchy sequences to Cauchy sequences.
- Proof. Consider $\epsilon > 0$.
- Then as f is uniformly continuous, there exists $\delta > 0$ such that

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Cauchy property

- Lemma 28.4: Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be uniformly continuous. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in A. Then $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
- In other words, uniformly continuous functions map Cauchy sequences to Cauchy sequences.
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- This proves that $\{f(x_n)\}$ is Cauchy.

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- END OF LECTURE 28.

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Lecture 29. Differentiation

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- More information on Madhava series: https://en.wikipedia.org/wiki/Madhava_series
- Here is link for more on ancient Indian mathematics: https://core.ac.uk/download/pdf/326681788.pdf

Differentiation

- Let $A \subseteq \mathbb{R}$. Fix $c \in A$. Assume that c is a cluster point of A. Let $f : A \to \mathbb{R}$ be a function. Then define $f_c : A \backslash \{c\} \to \mathbb{R}$ by
$$f_c(x) = \frac{f(x) - f(c)}{x - c}, \quad x \in A \backslash \{c\}.$$

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- Note that here f_c is not defined at c and we do not need it to consider this limit.
- More formally, we have the following definition.
- Definition 29.1: Let $A \subseteq \mathbb{R}$. Let $c \in A$ be a cluster point of A. Let $f : A \to \mathbb{R}$ be a function. Then f is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In such a case, $f'(c)$ is defined as this limit. If the limit does not exist f is said to be not differentiable at c.

Example

- Example 29.2 Let $f : [0,2] \to \mathbb{R}$ be the function

$$f(x) = x^3, \quad x \in [0,2].$$

Then f is differentiable at $c = 1$ and $f'(1) = 3$.

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- Proof: We have,
$$\begin{aligned}
\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} &= \lim_{x\to 1} \frac{x^3 - 1}{x - 1} \\
&= \lim_{x\to 1} \frac{(x-1)(x^2 + x + 1)}{x - 1} \\
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- Remark: We may also write $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ as
$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

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- Hence f is continuous at c.
- The function $g(x) = |x|, x \in \mathbb{R}$ is continuous at 0, but is not differentiable at 0 (Why?). ■

Algebra of differentiation

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$$(af + bg)'(c) = af'(c) + bg'(c).$$

- (ii) The product fg defined by $fg(x) = f(x)g(x), \quad x \in I$, is differentiable at c and

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- Proof. (i) The proof is clear.

Continuation

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$$\frac{f(x)g(x) - f(c)g(c)}{x - c} = \frac{f(x)(g(x) - g(c)) + (f(x) - f(c))g(c)}{x - c}$$
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$$\frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} = \frac{1}{g(x)g(c)} \frac{f(x)g(c) - f(c)g(x)}{x - c}$$
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- Now

$$\frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} = \frac{1}{g(x)g(c)} \frac{f(x)g(c) - f(c)g(x)}{x - c}$$
$$= \frac{1}{g(x)g(c)}[\frac{f(x) - f(c)}{x - c}.g(c) - \frac{f(c)(g(x) - g(c))}{x - c}]$$

- Now taking limit as x tends to c, we get

$$\frac{f}{g}'(c) = \frac{1}{g(c).g(c)}[f'(c)g(c) - f(c)g'(c)].$$

- That completes the proof.

Polynomials

- Theorem 29.5: Let $p : \mathbb{R} \to \mathbb{R}$ be a real polynomial:

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, x \in \mathbb{R}$$

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- Then at any $c \in \mathbb{R}$ p is differentiable at c and

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- Then at any $c \in \mathbb{R}$ p is differentiable at c and
$$p'(c) = a_1 + 2a_2 c + 3a_3 c^2 + \cdots + n a_n c^{(n-1)}.$$
- Proof. This can be proved using (i) and (ii) of previous theorem and induction. More directly:
$$\begin{aligned} & p'(c) \\ = \quad & \lim_{h \to 0} \frac{p(h+c) - p(h)}{h} \\ = \quad & \lim_{h \to 0} \frac{1}{h}[a_1.h + a_2((h+c)^2 - c^2)) + a_3(h+c)^3 - c^3) \\ & \quad + \cdots + a_n((h+c)^n - c^n) \\ = \quad & a_1 + 2a_2 c + 3a_3 c^2 + \cdots + n a_n c^{(n-1)}. \end{aligned}$$

Differentiable functions

- Definition 29.6: A function $f : I \to \mathbb{R}$ is said to be differentiable if it is differentiable at every $c \in I$. If $f : I \to \mathbb{R}$ is differentiable then the function $f' : I \to \mathbb{R}$ is called the first derivative of f.

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- END OF LECTURE 29.

- Definition 29.1: Let $A \subseteq \mathbb{R}$. Let $c \in A$ be a cluster point of A. Let $f : A \to \mathbb{R}$ be a function. Then f is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In such a case, $f'(c)$ is defined as this limit. If the limit does not exist f is said to be not differentiable at c.

Chain rule

- Theorem 30.1 Let I, J be intervals and let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be functions such that $f(I) \subseteq J$ and $h = g \circ f$. Consider $c \in I$. Suppose f is differentiable at c and g is differentiable at $f(c)$. Then h is differentiable at c and

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$$\frac{g \circ f(x) - g \circ f(c)}{x - c} = \frac{g \circ f(x) - g \circ f(c)}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}$$

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- Taking limit as x tends to c we should get the answer as $f(x)$ converges to $f(c)$.
- However, there is a problem here as we can't ensure that $f(x) - f(c) \neq 0$.

Caratheodory's theorem

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- Theorem 30.2: Let $f : I \to \mathbb{R}$ be a function where I is an interval. Fix $c \in I$. Then f is differentiable at c if and only if there exists a function $u : I \to \mathbb{R}$ such that

$$f(x) - f(c) = (x - c)u(x), \quad \forall x \in I \qquad (*)$$

and u is continuous at c. In such a case, $u(c) = f'(c)$.

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- Conversely if u exists satisfying $(*)$ and u is continuous at c
- From $(*)$, $u(x) = \frac{f(x)-f(c)}{x-c}$ for $x \neq c$. Taking limit as x tends to c, using continuity of u at c, f is differentiable at c, and $u(c) = f'(c)$. ■

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- Since $f(I) \subseteq J$, this equation is also true at $y = f(x)$ and so we get

$$g(f(x)) - g(f(c)) = (f(x) - f(c))v(f(x)), \quad \forall x \in I.$$

Continuation

- Now using the previous equation, we have

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- In other words $h'(c) = g'(f(c))f'(c)$. $\blacksquare$.

Derivative of inverse -I

- Theorem 30.3: Let I, J be intervals and let $f : I \to J$ be a bijection. Suppose f is differentiable at $c \in I$ and $g := f^{-1}$ is differentiable at $f(c)$. Then

$$g'(f(c)) = \frac{1}{f'(c)}.$$

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- Consequently, $g'(f(c)) = \frac{1}{f'(c)}$.■
- Note that this in particular means that in this Theorem, $f'(c) = 0$ is not possible.

Derivative of inverse -II

- Theorem 30.4: Let I, J be intervals and let $f : I \to J$ be a bijection. Suppose f is differentiable at $c \in I$ and $f'(c) \neq 0$. Also assume that f^{-1} is continuous at $f(c)$. Then $g := f^{-1}$ is differentiable at $f(c)$ and $g'(f(c)) = \frac{1}{f'(c)}$.

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- Now take $y = f(x)$ and $d = f(c)$ in the equation above, to get

$$y - d = (f^{-1}(x) - f^{-1}(d))u(f^{-1}(y))$$

Continuation

- Since f is surjective, this equation is true for every $y \in J$ and we get
$$g(y) - g(d) = (y - d)(\frac{1}{u(g(y))}).$$

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$$g(y) - g(d) = (y - d)\left(\frac{1}{u(g(y))}\right).$$
- Finally note that since $g = f^{-1}$ is continuous at d and u is continuous at c, $y \mapsto \frac{1}{u(g(y))}$ is continuous at d.

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- Therefore by Caratheodory's theorem g is differentiable at d, and the result follows.
- Example 30.5: For $n \in \mathbb{N}$ the function $g : (0, \infty) \to (0, \infty)$ defined by $g(y) = y^{\frac{1}{n}}$ is differentiable and
$$g'(y) = \frac{1}{ny^{1-\frac{1}{n}}}, \quad y \in (0, \infty).$$

Local extremums

- Definition 30.6: Let $f : I \to \mathbb{R}$ be a function and suppose $c \in I$. Then c is said to be a local maximum of f if there exists $\delta > 0$ such that

$$f(c) \geq f(x), \quad \forall x \in (c - \delta, c + \delta) \bigcap I.$$

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Interior extremum theorem

- Definition 30.8: Let I be an interval and let $c \in I$. Then c is said to be an interior point of I if there exists $\delta > 0$ such that

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Interior extremum theorem

- Definition 30.8: Let I be an interval and let $c \in I$. Then c is said to be an interior point of I if there exists $\delta > 0$ such that
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- Theorem 30.9: Let $f : I \to \mathbb{R}$ be a function. Suppose c is an interior point of I and suppose c is a local extremum of f. If f is differentiable at c then
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- So there exists $\delta_1 > 0$ such that $(c - \delta_1, c + \delta_1) \subseteq I$.
- Suppose that c is a local maximum of f. Then there exists $\delta_2 > 0$ such that
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Interior extremum theorem

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$$(c - \delta, c + \delta) \subseteq I.$$
- Theorem 30.9: Let $f : I \to \mathbb{R}$ be a function. Suppose c is an interior point of I and suppose c is a local extremum of f. If f is differentiable at c then
$$f'(c) = 0.$$
- Proof. Given that c is an interior point of f.
- So there exists $\delta_1 > 0$ such that $(c - \delta_1, c + \delta_1) \subseteq I$.
- Suppose that c is a local maximum of f. Then there exists $\delta_2 > 0$ such that
$$f(c) \geq f(x) \quad \forall x \in (c - \delta_2, c + \delta_2) \bigcap I.$$
- Taking $\delta = \min\{\delta_1, \delta_2\}$, we have $(c - \delta, c + \delta) \subseteq I$ and
$$f(c) \geq f(x), \quad \forall x \in (c - \delta, c + \delta).$$

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- Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $(c, c+\delta)$ converging to c (For instance, we can take $x_n = c + \frac{\delta}{2n}$.)
- Then for every n, $x_n > c$ and $f(x_n) \leq f(c)$ and hence

$$\frac{f(x_n) - f(c)}{x_n - c} \leq 0 \qquad (1)$$

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- Taking limit as $n \to \infty$, we get

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$$\frac{f(y_n) - f(c)}{y_n - c} \geq 0$$

- Taking limit as $n \to \infty$, we get

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- Combining inequalities (1) and (2) we get $f'(c) = 0$ as required. ■

Rolle's theorem

- Theorem 30.10 (Rolle's theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Suppose $f(a) = f(b) = 0$. Then there exists $c \in (a, b)$ such that

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- Proof: Since f is continuous on $[a, b]$, f attains global maximum and global minimum in $[a, b]$.
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- Similarly, if there exists $s \in (a, b)$ such that $f(s) < 0$ then global minimum is attained in (a, b) and if d is one such point, then $f'(d) = 0$.

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- Proof: Since f is continuous on $[a, b]$, f attains global maximum and global minimum in $[a, b]$.
- Suppose there exists some $t \in (a, b)$ such that $f(t) > 0$, then as $f(a) = f(b) = 0$, the global maximum of f is attained at some $c \in (a, b)$.
- In particular, c is a local extremum and by the interior extremum theorem, $f'(c) = 0$ and we are done.
- Similarly, if there exists $s \in (a, b)$ such that $f(s) < 0$ then global minimum is attained in (a, b) and if d is one such point, then $f'(d) = 0$.
- The only other possibility is $f(x) = 0$ for all $x \in [a, b]$ and in such a case $f'(x) = 0$ for all $x \in (a, b)$ and we are done. ■.

Example

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- END OF LECTURE 30

Lecture 31. Mean value theorem

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- Definition 29.1: Let $A \subseteq \mathbb{R}$. Let $c \in A$ be a cluster point of A. Let $f : A \to \mathbb{R}$ be a function. Then f is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In such a case, $f'(c)$ is defined as this limit. If the limit does not exist f is said to be not differentiable at c.

Interior Extremum theorem and Rolle's theorem

- Theorem 30.9 (Interior Extremum theorem): Let $f : I \to \mathbb{R}$ be a function. Suppose c is an interior point of I and suppose c is a local extremum of f. If f is differentiable at c then

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- If f is non-zero it attains either supremum or infimum at some interior point c in (a, b).
- Then by interior extremum theorem $f'(c) = 0$.

Mean value theorem (MVT)

- Theorem 31.1 (Mean value theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Then there exists $c \in (a, b)$ such that

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$$g(a) = g(b) = 0.$$

- Hence Rolle's theorem is applicable to g, and we get $c \in (a, b)$ such that $g'(c) = 0$.

Continuation

- Using linearity of differentiation,

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- Note that Rolle's theorem is a special case of mean value theorem.

Cauchy's mean value theorem

- Theorem 31.2 (Cauchy's Mean value theorem): Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions which are differentiable on (a, b). Then there exists $c \in (a, b)$ such that
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- Define $h : [a, b] \to \mathbb{R}$ by

$$h(x) = (f(b)-f(a))g(x)-f(x)(g(b)-g(a))-f(b)g(a)+f(a)g(b)$$

for $x \in [a, b]$.

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- Then h is continuous on $[a, b]$, differentiable on (a, b) and $h(a) = h(b) = 0$.

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- Therefore Rolle's theorem is applicable.

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- Then h is continuous on $[a, b]$, differentiable on (a, b) and $h(a) = h(b) = 0$.
- Therefore Rolle's theorem is applicable.
- So we get $c \in (a, b)$ such that $h'(c) = 0$ and that gives the result.
- Note that mean value theorem is a special case of Cauchy's mean value theorem with $g(x) = x, \quad x \in [a, b]$.

Applications of mean value theorem

- Corollary 31.3: Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on (a, b). Suppose $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant.

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- Clearly mean value theorem is applicable to this function and we get
$$f(t) - f(a) = 0.(t - a) = 0.$$
- Therefore $f(t) = f(a)$.
- In other words $f(t) = f(a)$ for every $t \in [a, b]$.■

Equal derivatives

- Corollary 31.4: Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions differentiable on (a, b). Suppose $f'(x) = g'(x)$ for all $x \in (a, b)$. Then $f(x) = g(x) + C, \quad x \in [a, b]$ for some $C \in \mathbb{R}$.

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- Proof: This is clear from the previous corollary, by considering the function, $h : [a, b] \to \mathbb{R}$ defined by

$$h(x) = f(x) - g(x), \quad x \in [a, b].$$

Monotonicity

- Recall that a function $f : [a, b] \to \mathbb{R}$ is said to be increasing (respectively decreasing) if $f(x) \leq f(y)$ (respectively $f(x) \geq f(y)$) for all x, y in $[a, b]$ with $x \leq y$.

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- Theorem 31.5: Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b).

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- Theorem 31.5: Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b).
- (i) f is increasing on $[a, b]$ if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

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- Theorem 31.5: Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b).
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- (ii) f is decreasing on $[a, b]$ if and only if $f'(x) \leq 0$ for all $x \in (a, b)$.
- Proof: (i) Suppose f is increasing and $x \in (a, b)$.

Monotonicity

- Recall that a function $f : [a, b] \to \mathbb{R}$ is said to be increasing (respectively decreasing) if $f(x) \leq f(y)$ (respectively $f(x) \geq f(y)$) for all x, y in $[a, b]$ with $x \leq y$.
- Theorem 31.5: Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b).
- (i) f is increasing on $[a, b]$ if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.
- (ii) f is decreasing on $[a, b]$ if and only if $f'(x) \leq 0$ for all $x \in (a, b)$.
- Proof: (i) Suppose f is increasing and $x \in (a, b)$.
- Consider any sequence $\{x_n\}$ in (a, b) with $x < x_n \leq b$, converging to x. Then $f(x_n) - f(x) \geq 0$ for all n and we get

$$f'(x) = \lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} \geq 0.$$

Continuation

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- Proof of (ii) is similar. ■

Strictly increasing functions

- Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose $f'(x) > 0$ for all $x \in (a, b)$ then by mean value theorem it is easy to see that f is strictly increasing.

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- Remark 31.7: In this Example, 0 is a point which is never picked up by the mean value theorem. That is, for no $x, y \in [-1, 1]$ with $x < y$, $f(y) - f(x) = f'(0)(y - x)$. Can we characterize such points?

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- END OF LECTURE 31.

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- Theorem 30.10 (Rolle's theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Suppose $f(a) = f(b) = 0$. Then there exists $c \in (a, b)$ such that

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- Theorem 31.1 (Mean value theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Then there exists $c \in (a, b)$ such that

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Continuation

- Note that, for $a \leq x_0 \leq x \leq b$, by considering f restricted to $[x_0, x]$, by mean value theorem we get

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$$f(x) = f(x_0) + f'(c)(x - x_0).$$

- Taylor's theorem gives similar result for higher order derivatives.

Higher derivatives

- We recall a few definitions.
- Definition 29.6: A function $f : I \to \mathbb{R}$ is said to be differentiable if it is differentiable at every $c \in I$. If $f : I \to \mathbb{R}$ is differentiable then the function $f' : I \to \mathbb{R}$ is called the first derivative of f.

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- f is said to be infinitely differentiable if it has n-th derivative for every $n \in \mathbb{N}$.
- We can see that polynomials are infinitely differentiable.

Taylor's polynomial

- Definition 32.1: Let $f : [a, b] \to \mathbb{R}$ be a function. Fix $x_0 \in [a, b]$. Assume $f^{(1)}(x_0), f^{(2)}(x_0), \ldots, f^{(n)}(x_0)$ exist. Then the polynomial P_n defined by $P_n(x) =$

$$f(x_0)+f^{(1)}(x_0)(x-x_0)+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

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- Given f as above, we wish to say that P_n approximates f. We write $R_n(x) = f(x) - P_n(x), \quad x \in [a, b]$ or equivalently,

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- Here R_n is known as the remainder term or the error term. The main problem here is to get a suitable formula for R_n and to estimate it.

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- Theorem 32.3 (Taylor's theorem): Let $f : [a, b] \to \mathbb{R}$ be a function. Fix $x_0 \in [a, b]$. Suppose for some $n \in \mathbb{N}$, $f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ exist and are continuous on $[a, b]$, and further $f^{(n+1)}$ exists on (a, b). Then for any $x \in [a, b]$, there exists c strictly in between x_0 and x such that

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- Here c is in (x_0, x) if $x_0 < x$ and it is in (x, x_0) if $x < x_0$. If $x = x_0$, the equation above is a triviality for any c.

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- Moreover, for every $t \in (x_0, x)$

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- Consider $g : [x_0, x] \to \mathbb{R}$ defined by

$$g(t) = h(t) - \left(\frac{x - t}{x - x_0}\right)^{(n+1)} h(x_0), \quad t \in [x_0, x].$$

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- Equivalently,

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- Then g is continuous on $[x_0, x]$, differentiable on (x_0, x). Moreover, $g(x_0) = 0$ and $g(x) = h(x) = 0$.
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- By Rolle's theorem, there exists $c \in (x_0, x)$ such that $g'(c) = 0$.
- Hence using equation (1),
$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)(\frac{x-c}{x-x_0})^n h(x_0) = 0.$$
- Equivalently,
$$h(x_0) = \frac{(x-x_0)^n}{(n+1)!} f^{(n+1)}(c).$$
- This is the formula for the remainder term we wanted to obtain. ■

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$$\begin{aligned} f'(x) &\geq 0, \quad \forall x \in (c - \delta, c); \\ f'(x) &\leq 0, \quad \forall x \in (c, c + \delta). \end{aligned}$$

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- (ii) Assume that there exists $\delta > 0$ such that f is differentiable on $(c - \delta, c) \bigcup (c, c + \delta)$ and

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Continuation

- Proof: (i) For $c - \delta < x < c$, by considering f restricted to $[x, c]$, by mean value theorem we get

$$f(c) - f(x) = f'(d)(c - x)$$

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- Proof: (i) For $c-\delta < x < c$, by considering f restricted to $[x, c]$, by mean value theorem we get

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- for some $d \in (x, c) \subseteq (c-\delta, c)$. Hence $f'(d) \geq 0$. So $f(c) - f(x) \geq 0$ or $f(c) \geq f(x)$.

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- Similarly, if $x \in (c, c + \delta)$, consider f restricted to $[c, x]$. By mean value theorem,

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- The proof of (ii) is similar. ■.

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- (ii) If n is even and $f^{(n)}(x_0) < 0$ then f has local maximum at x_0.
- (iii) If n is odd then f has neither local maximum nor local minimum at x_0.

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$$f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x - x_0)^n$$

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- (iii) Now as n is odd, $(x - x_0)^n$ is either positive or negative depending upon $x > x_0$ or $x < x_0$.

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- and we see that $f(x) - f(x_0)$ has different signs for $x > x_0$ and $x < x_0$ in an open interval around x_0. $\blacksquare$.

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Lecture 33. L'Hospital's rules

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- Theorem 30.10 (Rolle's theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Suppose $f(a) = f(b) = 0$. Then there exists $c \in (a, b)$ such that

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- Theorem 31.1 (Mean value theorem): Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Then there exists $c \in (a, b)$ such that

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- Observe that,

$$\lim_{x \to c+} f(x) = z$$

  iff for every decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in A converging to c, $\{f(x_n)\}$ converges to z.
- Some texts may have the notation: $\lim_{x \downarrow c} f(x) = z$.

L'Hospital's rule -0

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- Theorem 33.2 (L'Hospital's rule I (a):) Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions. Suppose $g'(x) \neq 0$ for every $x \in (a, b)$. Assume

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- For $\epsilon > 0$, choose $\delta > 0$ such that

$$|\frac{f'(x)}{g'(x)} - L| < \epsilon$$

for $a < x < a + \delta$.

Continuation

- Now for any $a < y < x < a + \delta$, by Cauchy's mean value theorem

$$(f(x) - f(y))g'(c) = f'(c)(g(x) - g(y))$$

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  for all $x \in (a, a + \delta)$.
- This proves that

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L.$$

L'Hospital's rule I(b)

- **Theorem 33.2 (L'Hospital's rule I (b)):** Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions. Suppose $g'(x) \neq 0$ for every $x \in (a, b)$. Assume

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- If $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \{+\infty, -\infty\}$ then

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- Now for $M \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\frac{f'(x)}{g'(x)} > M$$

for $x \in (a, a + \delta)$.

Continuation

- By Cauchy's mean value theorem, for $a < y < x < a + \delta$,

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- Like before, $g'(c) \neq 0$, $g(x) - g(y) \neq 0$ and we get

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- This absurdity shows that we should give a 'sensible meaning' to $\sum_{n=1}^{\infty} a_n$.

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- Example 3 (Harmonic series).

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent, as } \left\{ \sum_{k=1}^{n} \frac{1}{k} \right\}_{n\in\mathbb{N}} \text{ is not bounded above.}$$

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  where $\{t_n\}_{n\in\mathbb{N}}$ is the sequence of partial sums of $\sum_{n=1}^{\infty}(\frac{1}{2})^{n-1}$.

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- Theorem 1 (Cauchy criterion). An infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if for every $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon, \quad \forall m > n \geq K.$$

- Theorem 2 (n^{th} term test). If a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Algebra of convergent series

Theorem 3. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series with sums x and y, respectively. Then

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- So, given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, one may think of defining their product as $\sum_{n=1}^{\infty} c_n$, where $c_n = a_n b_n$.

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then their product is a polynomial

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where $c_0 = a_0 b_0$, $c_1 = a_0 b_1 + a_1 b_0$, $c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$, and in general

$$c_n = a_0 b_n + a_1 n_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.$$

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- This suggests the following definition.

- Definition 2. Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their Cauchy product is the series $\sum_{n=0}^{\infty} c_n$, where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$.

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  Consider the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, where

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Then $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent by the following result.
(**Result:** The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, where $\{a_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of positive reals, is convergent if and only if $\lim_{n \to \infty} a_n = 0$.)

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- However, things are not that bad. We will revisit this and see when can we assure that the Cauchy product of two series is convergent.

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Proof: Exercise

- **Theorem 5 (Comparison test).** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be real sequences, and suppose that there exists $N \in \mathbb{N}$ such that

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Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

- Exercise: A series $\sum_{k=1}^{\infty} b_n$ is said to be a telescoping series if there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $b_n = a_{n+1} - a_n$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\lim_{n\to\infty} a_n$ exists. In such a case, find the sum.

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- Theorem 6 (Limit comparison test): Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be strictly positive sequences.
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Infinite Series L2. Recall

- Definition. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

  An expression of the form $\sum_{n=1}^{\infty} a_n$ is called an infinite series.

  For each $n \in \mathbb{N}$, the finite sum $s_n = \sum_{k=1}^{n} a_k$ is called the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$.

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Since $\epsilon > 0$ is arbitrary, Cauchy criterion implies that $\sum_{n=1}^{\infty} a_n$ is convergent. ■

Tests for absolute convergence

- Theorem (Cauchy's Root Test). Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence.
  - (i) If there exist $r \in \mathbb{R}$ with $r < 1$ and $K \in \mathbb{N}$ such that

$$|a_n|^{\frac{1}{n}} \leq r, \ \forall n \geq K, \tag{1}$$

then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

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  Therefore, by n^{th} term test, the series $\sum_{n=1}^{\infty} a_n$ is divergent. ■

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Since $s > 1$, by (ii) of the previous theorem, we get that $\sum_{n=1}^{\infty} a_n$ is divergent. ∎

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Proof: (i) Since (4) holds, we have $|a_{n+1}| \leq r|a_n|, \ \forall n \geq K$.

$$\implies \quad |a_{n+K}| \leq |a_K| r^n = \frac{|a_K|}{r^K} r^{n+K}, \ \forall n \in \mathbb{N}.$$

$$\implies \quad |a_n| \leq \frac{|a_K|}{r^K} r^n, \ \forall n \geq K+1.$$

Now, since $r < 1$, the geometric series $\sum_{n=1}^{\infty} r^n$ is convergent. This implies that the series $\sum_{n=1}^{\infty} \frac{|a_K|}{r^K} r^n$ is convergent. Therefore, by comparison test, $\sum_{n=1}^{\infty} |a_n|$ is convergent.

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  Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonzero real numbers and suppose that

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  Proof: Exercise.

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- The answer is NO, as seen from the next result.

Convergence of Cauchy product

- Theorem (Mertens' Theorem). Let $\sum_{n=0}^{\infty} a_n$ be absolutely convergent and $\sum_{n=0}^{\infty} b_n$ be convergent. If $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = b$, then their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent and $\sum_{n=0}^{\infty} c_n = ab$.

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  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$, $\{t_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, and $\sum_{n=0}^{\infty} c_n$, respectively.

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$$
\begin{aligned}
u_n &= c_0 + c_1 + \cdots + c_n \\
&= (a_0 b_0) + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\
&= a_0(b_0 + \cdots + b_n) + a_1(b_0 + \cdots + b_{n-1}) + \cdots + a_n b_0 \\
&= a_0 t_n + a_1 t_{n-1} + \cdots + a_n t_0 \\
&= a_0 t_n + a_1 t_{n-1} + \cdots + a_n t_0 - \left(\sum_{k=0}^{n} a_k \right) b + s_n b \\
&= a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b) + s_n b,
\end{aligned}
$$

i.e.,

$$\begin{aligned} c_n &= a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b) + s_n b \\ &= v_n + s_n b, \end{aligned} \tag{7}$$

where $v_n = a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b)$ for all $n \in \mathbb{N} \cup \{0\}$.

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Since $\{t_n - b\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded, there exists $M > 0$ such that

$$|t_n - b| \leq M, \ \forall n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, say $\sum_{n=1}^{\infty} |a_n| = \alpha$,by Cauchy criterion there exists $K_2 \in \mathbb{N}$ such that

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|v_n| &= |a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b)| \\
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&= |a_0||t_n - b| + |a_1||t_{n-1} - b| + \cdots + |a_{n-K}||t_{n+K} - b| \\
&\quad + |a_{n-K+1}||t_{n+K-1} - b| + \cdots + |a_n||t_0 - b| \\
&\leq (|a_0| + |a_1| + \cdots + |a_{n-K}|)\epsilon \\
&\quad + (|a_{n-K+1}| + \cdots + |a_n|)M \\
&\leq \alpha\epsilon + \epsilon M \\
&= (\alpha + M)\epsilon.
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&\quad + (|a_{n-K+1}| + \cdots + |a_n|)M \\
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Recall

- Definition. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that $\sum_{n=1}^{\infty} a_n$ is
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- Theorem (Cauchy's Root Test).
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- **Remark.** The Cauchy product of two convergent series need not be convergent.

Infinite Series L3. Convergence of Cauchy product

- **Theorem (Mertens' Theorem).** Let $\sum_{n=0}^{\infty} a_n$ be absolutely convergent and $\sum_{n=0}^{\infty} b_n$ be convergent. If $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = b$, then their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent and $\sum_{n=0}^{\infty} c_n = ab$.

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  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$, $\{t_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, and $\sum_{n=0}^{\infty} c_n$, respectively.

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$$
\begin{aligned}
u_n &= c_0 + c_1 + \cdots + c_n \\
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&= a_0(b_0 + \cdots + b_n) + a_1(b_0 + \cdots + b_{n-1}) + \cdots + a_n b_0 \\
&= a_0 t_n + a_1 t_{n-1} + \cdots + a_n t_0 \\
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- Theorem (Alternating Series Test). Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive reals such that $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

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$$\sum_{k=n+1}^{m} a_k b_k = (a_m s_m - a_{n+1} s_n) + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) s_k. \quad (9)$$

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Theorem (Abel's Test). Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent monotone sequence and let the series $\sum_{n=1}^{\infty} b_n$ be convergent. Then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

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Theorem (Abel's Test). Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent monotone sequence and let the series $\sum_{n=1}^{\infty} b_n$ be convergent. Then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

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This implies by (11) that the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent, because by hypothesis $\sum_{n=1}^{\infty} b_n$ is convergent.

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(ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$ is convergent by Abel's test.

Grouping of series

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  Proof: Exercise

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$$\begin{aligned} t_{3n} &= \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}\right) + \cdots \\ &= \left(\frac{1}{2} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{4n-2} - \frac{1}{4n}\right) + \cdots \\ &= \frac{s_{2n}}{2} \to \frac{s}{2} \end{aligned}$$

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- Definition. A series $\sum_{n=1}^{\infty} b_n$ is said to be a rearrangement of a series $\sum_{n=1}^{\infty} a_n$ if there is a bijection f of $\mathbb{N}$ onto $\mathbb{N}$ such that $b_k = a_{f(k)}$ for all $k \in \mathbb{N}$.

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- However, things are not that bad when we deal with absolutely convergent series.

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Since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{n\to\infty} t_n = a$. ■.

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- This theorem should convince us of the danger of manipulating an infinite series without any attention to rigorous analysis.
- To prove this theorem, we need the notions of positive and negative parts of a series.

- Given a series $\sum_{n=1}^{\infty} a_n$, let

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- Note that all the terms of both these series are non-negative.
- For example, if $a_n = \frac{(-1)^{n+1}}{n}$, then

$$\sum_{n=1}^{\infty} a_n^+ = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots$$

and

$$\sum_{n=1}^{\infty} a_n^- = 0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + \cdots$$

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  By hypothesis $\sum_{n=1}^{\infty} |a_n|$ is divergent, which implies that $\lim_{n\to\infty} t_n = \infty$.

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- Reference: Theorem 3.54 in [Walter Rudin, Principles of Mathematical Analysis, Third Edition, McGraw Hill Inc., 1976]

  or

  Theorem 8.33 in [Tom M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, Inc., 1974]

Infinite products

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- By analogy with infinite series, it seems natural to call the product $\prod_{n=1}^{\infty} a_n$ converges if $\{p_n\}_{n\in\mathbb{N}}$ converges.
- However, this definition is inconvenient since every product having one factor zero would converge regardless of the behavior of the other factors.

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  (ii) If no factor a_n is zero, then we say that the product is convergent if there exists a real number $p \neq 0$ such that $\lim_{n\to\infty} p_n = p$.
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  (iii) If there exists an $N \in \mathbb{N}$ such that $n > N$ implies $a_n \neq 0$, then we say that $\prod_{n=1}^{\infty} a_n$ is convergent provided that $\prod_{n=N+1}^{\infty} a_n$ converges as described in (ii).

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  (iii) If there exists an $N \in \mathbb{N}$ such that $n > N$ implies $a_n \neq 0$, then we say that $\prod_{n=1}^{\infty} a_n$ is convergent provided that $\prod_{n=N+1}^{\infty} a_n$ converges as described in (ii).
  In this case the value of the product $\prod_{n=1}^{\infty} a_n$ is
  $$a_1 a_2 \cdots a_N \prod_{n=N+1}^{\infty} a_n.$$
  (iv) $\prod_{n=1}^{\infty} a_n$ is called divergent if it does not converge as described in (ii) or (iii).

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- Theorem (Cauchy criterion). The infinite product $\prod_{n=1}^{\infty} a_n$ is convergent if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

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- Theorem. If $\prod_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 1$.
- For this reason, the factors of a product are written as $1 + a_n$ instead of just a_n. Thus, if $\prod_{n=1}^{\infty}(1 + a_n)$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

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- Reference: pp. 206-209 of [Tom M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, Inc., 1974]