ANALYSIS-I

Chaitanya G K

Indian Statistical Institute, Bangalore

Infinite Series

- We know that finite sums $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ of real numbers are well-defined due to the associativity of addition.

Infinite Series

- We know that finite sums $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ of real numbers are well-defined due to the associativity of addition.
- Now, it is natural to ask: What is the meaning of $\sum_{n=1}^{\infty} a_n$ when $\{a_n\}_{n\in\mathbb{N}}$ is a real sequence?

Infinite Series

- We know that finite sums $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ of real numbers are well-defined due to the associativity of addition.
- Now, it is natural to ask: What is the meaning of $\sum_{n=1}^{\infty} a_n$ when $\{a_n\}_{n\in\mathbb{N}}$ is a real sequence?
- For example, consider $\sum_{n=1}^{\infty} a_n$ with $a_n = (-1)^{n+1}, \forall n \in \mathbb{N}$,

$$\text{i.e., consider the sum} \quad 1 - 1 + 1 - 1 + \cdots .$$

Infinite Series

- We know that finite sums $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ of real numbers are well-defined due to the associativity of addition.
- Now, it is natural to ask: What is the meaning of $\sum_{n=1}^{\infty} a_n$ when $\{a_n\}_{n\in\mathbb{N}}$ is a real sequence?
- For example, consider $\sum_{n=1}^{\infty} a_n$ with $a_n = (-1)^{n+1}, \forall n \in \mathbb{N}$,

  i.e., consider the sum $\quad 1 - 1 + 1 - 1 + \cdots .$

-

$$
\begin{aligned}
1 - 1 + 1 - 1 + \cdots &= (1-1) + (1-1) + \cdots \\
&= 0 + 0 + \cdots = 0 \\
1 - 1 + 1 - 1 + \cdots &= 1 + (-1+1) + (-1+1) + \cdots \\
&= 1 + 0 + 0 + \cdots = 1
\end{aligned}
$$

Infinite Series

- We know that finite sums $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ of real numbers are well-defined due to the associativity of addition.
- Now, it is natural to ask: What is the meaning of $\sum_{n=1}^{\infty} a_n$ when $\{a_n\}_{n\in\mathbb{N}}$ is a real sequence?
- For example, consider $\sum_{n=1}^{\infty} a_n$ with $a_n = (-1)^{n+1}, \forall n \in \mathbb{N}$,

$$\text{i.e., consider the sum} \quad 1 - 1 + 1 - 1 + \cdots .$$

-

$$\begin{aligned} 1 - 1 + 1 - 1 + \cdots &= (1-1) + (1-1) + \cdots \\ &= 0 + 0 + \cdots = 0 \\ 1 - 1 + 1 - 1 + \cdots &= 1 + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + \cdots = 1 \end{aligned}$$

- This absurdity shows that we should give a 'sensible meaning' to $\sum_{n=1}^{\infty} a_n$.

Convergence and Sum of an infinite series

- Definition 1: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

  An expression of the form $\sum_{n=1}^{\infty} a_n$ is called an infinite series.

Convergence and Sum of an infinite series

- Definition 1: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

  An expression of the form $\sum_{n=1}^{\infty} a_n$ is called an infinite series.

  For each $n \in \mathbb{N}$, the finite sum $s_n = \sum_{k=1}^{n} a_k$ is called the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$.

Convergence and Sum of an infinite series

- Definition 1: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

  An expression of the form $\sum_{n=1}^{\infty} a_n$ is called an infinite series.

  For each $n \in \mathbb{N}$, the finite sum $s_n = \sum_{k=1}^{n} a_k$ is called the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$.

  The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if $\{s_n\}_{n\in\mathbb{N}}$ is convergent.

  In such a case, the limit $s := \lim_{n\to\infty} s_n$ is called the sum of the series, and we denote this fact by the symbol $\sum_{n=1}^{\infty} a_n = s$.

Convergence and Sum of an infinite series

- Definition 1: Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

  An expression of the form $\sum_{n=1}^{\infty} a_n$ is called an infinite series.

  For each $n \in \mathbb{N}$, the finite sum $s_n = \sum_{k=1}^{n} a_k$ is called the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$.

  The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if $\{s_n\}_{n\in\mathbb{N}}$ is convergent.

  In such a case, the limit $s := \lim_{n\to\infty} s_n$ is called the sum of the series, and we denote this fact by the symbol $\sum_{n=1}^{\infty} a_n = s$.

  The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be divergent if $\{s_n\}_{n\in\mathbb{N}}$ is divergent.

Examples

- Example 1. Consider the infinite series $\sum_{n=1}^{\infty} a_n$, where $a_n = c, \forall n \in \mathbb{N}$.

Examples

- Example 1. Consider the infinite series $\sum_{n=1}^{\infty} a_n$, where $a_n = c, \forall n \in \mathbb{N}$.

  Then $\sum_{n=1}^{\infty} a_n$ is convergent $\iff c = 0$.

Examples

- Example 1. Consider the infinite series $\sum_{n=1}^{\infty} a_n$, where $a_n = c, \forall n \in \mathbb{N}$.

  Then $\sum_{n=1}^{\infty} a_n$ is convergent $\iff c = 0$.

  (In fact, $\{s_n\}_{n\in\mathbb{N}} = \{nc\}_{n\in\mathbb{N}}$ is convergent $\iff c = 0$)

Examples

- Example 1. Consider the infinite series $\sum_{n=1}^{\infty} a_n$, where $a_n = c, \forall n \in \mathbb{N}$.

  Then $\sum_{n=1}^{\infty} a_n$ is convergent $\iff c = 0$.

  (In fact, $\{s_n\}_{n\in\mathbb{N}} = \{nc\}_{n\in\mathbb{N}}$ is convergent $\iff c = 0$)

- Example 2 (Geometric series).

$$1 + r + r^2 + \cdots = \frac{1}{1-r} \text{ for } |r| < 1.$$

Examples

- Example 1. Consider the infinite series $\sum_{n=1}^{\infty} a_n$, where $a_n = c, \forall n \in \mathbb{N}$.

  Then $\sum_{n=1}^{\infty} a_n$ is convergent $\iff c = 0$.

  (In fact, $\{s_n\}_{n\in\mathbb{N}} = \{nc\}_{n\in\mathbb{N}}$ is convergent $\iff c = 0$)

- Example 2 (Geometric series).

$$1 + r + r^2 + \cdots = \frac{1}{1-r} \text{ for } |r| < 1.$$

  (In fact, $s_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r} \to \frac{1}{1-r}$ for $|r| < 1$)

Examples

- Example 1. Consider the infinite series $\sum_{n=1}^{\infty} a_n$, where $a_n = c, \forall n \in \mathbb{N}$.

  Then $\sum_{n=1}^{\infty} a_n$ is convergent $\iff c = 0$.

  (In fact, $\{s_n\}_{n\in\mathbb{N}} = \{nc\}_{n\in\mathbb{N}}$ is convergent $\iff c = 0$)

- Example 2 (Geometric series).

$$1 + r + r^2 + \cdots = \frac{1}{1-r} \text{ for } |r| < 1.$$

  (In fact, $s_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r} \to \frac{1}{1-r}$ for $|r| < 1$)

- Example 3 (Harmonic series).

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent, as } \left\{ \sum_{k=1}^{n} \frac{1}{k} \right\}_{n\in\mathbb{N}} \text{ is not bounded above.}$$

- Example 4.
  The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

- Example 4.

  The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

- Example 4.
  The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Look at the subsequence $\{s_{2^n-1}\}_{n\in\mathbb{N}}$.

- Example 4.

  The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Look at the subsequence $\{s_{2^n-1}\}_{n\in\mathbb{N}}$.

$$s_{2^1-1} = 1$$

- Example 4.
  The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Look at the subsequence $\{s_{2^n-1}\}_{n\in\mathbb{N}}$.

$$
\begin{aligned}
s_{2^1-1} &= 1 \\
s_{2^2-1} &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) \leq 1 + \left(\frac{2}{2^2}\right) = 1 + \frac{1}{2}
\end{aligned}
$$

- Example 4.
  The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Look at the subsequence $\{s_{2^n-1}\}_{n\in\mathbb{N}}$.

$$
\begin{aligned}
s_{2^1-1} &= 1 \\
s_{2^2-1} &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) \leq 1 + \left(\frac{2}{2^2}\right) = 1 + \frac{1}{2} \\
s_{2^3-1} &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right)
\end{aligned}
$$

- **Example 4.**
  The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Look at the subsequence $\{s_{2^n-1}\}_{n\in\mathbb{N}}$.

$$
\begin{aligned}
s_{2^1-1} &= 1 \\
s_{2^2-1} &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) \le 1 + \left(\frac{2}{2^2}\right) = 1 + \frac{1}{2} \\
s_{2^3-1} &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) \\
&\le 1 + \frac{1}{2} + \left(\frac{4}{4^2}\right) = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2
\end{aligned}
$$

- **Example 4.**
  The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Look at the subsequence $\{s_{2^n-1}\}_{n\in\mathbb{N}}$.

$$
\begin{aligned}
s_{2^1-1} &= 1 \\
s_{2^2-1} &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) \le 1 + \left(\frac{2}{2^2}\right) = 1 + \frac{1}{2} \\
s_{2^3-1} &= 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) \\
&\le 1 + \frac{1}{2} + \left(\frac{4}{4^2}\right) = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 \\
&\vdots \\
s_{2^n-1} &\le 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1} =: t_n, \quad \forall n \in \mathbb{N},
\end{aligned}
$$

  where $\{t_n\}_{n\in\mathbb{N}}$ is the sequence of partial sums of $\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}$.

$$\implies s_{2^n-1} \leq t_n \leq 2, \quad \forall n \in \mathbb{N}.$$

$$\implies s_{2^n-1} \leq t_n \leq 2, \quad \forall n \in \mathbb{N}.$$

$$\implies s_n \leq 2, \quad \forall n \in \mathbb{N} \quad \text{(why?)}.$$

$$\implies s_{2^n-1} \le t_n \le 2, \quad \forall n \in \mathbb{N}.$$

$$\implies s_n \le 2, \quad \forall n \in \mathbb{N} \quad \text{(why?)}.$$

Therefore $\{s_n\}_{n\in\mathbb{N}}$ is convergent, being an increasing sequence.

$$\implies s_{2^n-1} \leq t_n \leq 2, \quad \forall n \in \mathbb{N}.$$

$$\implies s_n \leq 2, \quad \forall n \in \mathbb{N} \ \text{(why?)}.$$

Therefore $\{s_n\}_{n\in\mathbb{N}}$ is convergent, being an increasing sequence.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. $\square$

$$\implies s_{2^n-1} \leq t_n \leq 2, \quad \forall n \in \mathbb{N}.$$

$$\implies s_n \leq 2, \quad \forall n \in \mathbb{N} \quad \text{(why?)}.$$

Therefore $\{s_n\}_{n\in\mathbb{N}}$ is convergent, being an increasing sequence.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. $\square$

- Note: In fact, it can be proved that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (see Basel problem).

$$\implies s_{2^n-1} \le t_n \le 2, \quad \forall n \in \mathbb{N}.$$

$$\implies s_n \le 2, \quad \forall n \in \mathbb{N} \quad \text{(why?)}.$$

Therefore $\{s_n\}_{n\in\mathbb{N}}$ is convergent, being an increasing sequence.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. □

- Note: In fact, it can be proved that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (see Basel problem).
- Exercise: Prove that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for all $p \in \mathbb{N} \setminus \{1\}$.

$$\implies s_{2^n-1} \le t_n \le 2, \quad \forall n \in \mathbb{N}.$$

$$\implies s_n \le 2, \quad \forall n \in \mathbb{N} \quad \text{(why?)}.$$

Therefore $\{s_n\}_{n\in\mathbb{N}}$ is convergent, being an increasing sequence.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. □

- Note: In fact, it can be proved that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (see Basel problem).

- Exercise: Prove that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for all $p \in \mathbb{N} \setminus \{1\}$.

- Theorem 1 (Cauchy criterion). An infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if for every $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon, \quad \forall m > n \ge K.$$

- Theorem 2 (n^{th} term test). If a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Algebra of convergent series

Theorem 3. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series with sums x and y, respectively. Then

(i) $\sum_{n=1}^{\infty}(a_n + b_n) = x + y$;

(ii) $\sum_{n=1}^{\infty}(ca_n) = cx$ for all $c \in \mathbb{R}$.

Algebra of convergent series

Theorem 3. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series with sums x and y, respectively. Then

(i) $\sum_{n=1}^{\infty}(a_n + b_n) = x + y$;

(ii) $\sum_{n=1}^{\infty}(ca_n) = cx$ for all $c \in \mathbb{R}$.

Sketch of the proof:

(i) Let $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, respectively.

Algebra of convergent series

Theorem 3. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series with sums x and y, respectively. Then

(i) $\sum_{n=1}^{\infty}(a_n + b_n) = x + y$;

(ii) $\sum_{n=1}^{\infty}(ca_n) = cx$ for all $c \in \mathbb{R}$.

Sketch of the proof:

(i) Let $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, respectively.

Similarly, let $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty}(a_n + b_n)$.

Algebra of convergent series

Theorem 3. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series with sums x and y, respectively. Then

(i) $\sum_{n=1}^{\infty}(a_n + b_n) = x + y$;

(ii) $\sum_{n=1}^{\infty}(ca_n) = cx$ for all $c \in \mathbb{R}$.

Sketch of the proof:

(i) Let $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, respectively.

Similarly, let $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty}(a_n + b_n)$.

Then

$$u_n = \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = s_n + t_n \to x + y \text{ as } n \to \infty.$$

Algebra of convergent series

Theorem 3. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series with sums x and y, respectively. Then

(i) $\sum_{n=1}^{\infty}(a_n + b_n) = x + y$;

(ii) $\sum_{n=1}^{\infty}(ca_n) = cx$ for all $c \in \mathbb{R}$.

Sketch of the proof:

(i) Let $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, respectively.

Similarly, let $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty}(a_n + b_n)$.

Then

$$u_n = \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = s_n + t_n \to x + y \text{ as } n \to \infty.$$

(ii) Similar $\square$

Cauchy product

- Now it is natural to ask: Does the ‘product’ of two convergent series is convergent?

Cauchy product

- Now it is natural to ask: Does the 'product' of two convergent series is convergent?
- Recall that given two convergent sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{a_b\}_{n\in\mathbb{N}}$, we defined their product as the sequence $\{a_nb_n\}_{n\in\mathbb{N}}$ and the product converges to the product $(\lim_{n\to\infty} a_n)\cdot(\lim_{n\to\infty} b_n)$.

Cauchy product

- Now it is natural to ask: Does the 'product' of two convergent series is convergent?
- Recall that given two convergent sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{a_b\}_{n\in\mathbb{N}}$, we defined their product as the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$ and the product converges to the product $(\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
- So, given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, one may think of defining their product as $\sum_{n=1}^{\infty} c_n$, where $c_n = a_n b_n$.

Cauchy product

- Now it is natural to ask: Does the 'product' of two convergent series is convergent?
- Recall that given two convergent sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{a_b\}_{n\in\mathbb{N}}$, we defined their product as the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$ and the product converges to the product $(\lim_{n\to\infty} a_n)\cdot(\lim_{n\to\infty} b_n)$.
- So, given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, one may think of defining their product as $\sum_{n=1}^{\infty} c_n$, where $c_n = a_n b_n$.
- But, this is not a good definition.

Cauchy product

- Now it is natural to ask: Does the 'product' of two convergent series is convergent?
- Recall that given two convergent sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{a_b\}_{n\in\mathbb{N}}$, we defined their product as the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$ and the product converges to the product $(\lim_{n\to\infty} a_n)\cdot(\lim_{n\to\infty} b_n)$.
- So, given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, one may think of defining their product as $\sum_{n=1}^{\infty} c_n$, where $c_n = a_n b_n$.
- But, this is not a good definition.
- In fact, even for $n = 2$, the equality $(a_1 + a_2)(b_1 + b_2) = a_1 b_1 + a_2 b_2$ is not true in general.

Cauchy product

- Now it is natural to ask: Does the 'product' of two convergent series is convergent?
- Recall that given two convergent sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{a_b\}_{n\in\mathbb{N}}$, we defined their product as the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$ and the product converges to the product $(\lim_{n\to\infty} a_n)\cdot(\lim_{n\to\infty} b_n)$.
- So, given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, one may think of defining their product as $\sum_{n=1}^{\infty} c_n$, where $c_n = a_n b_n$.
- But, this is not a good definition.
- In fact, even for $n = 2$, the equality $(a_1 + a_2)(b_1 + b_2) = a_1 b_1 + a_2 b_2$ is not true in general.
- Recall that we have used distributivity while computing $(a_1 + a_2)(b_1 + b_2)$

Cauchy product

- Now it is natural to ask: Does the 'product' of two convergent series is convergent?
- Recall that given two convergent sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{a_b\}_{n\in\mathbb{N}}$, we defined their product as the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$ and the product converges to the product $(\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
- So, given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, one may think of defining their product as $\sum_{n=1}^{\infty} c_n$, where $c_n = a_n b_n$.
- But, this is not a good definition.
- In fact, even for $n = 2$, the equality $(a_1 + a_2)(b_1 + b_2) = a_1 b_1 + a_2 b_2$ is not true in general.
- Recall that we have used distributivity while computing $(a_1 + a_2)(b_1 + b_2)$
- Indeed $(a_1 + a_2)(b_1 + b_2) = a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$

- Can we use distributivity for an infinite sum?

- Can we use distributivity for an infinite sum?
- If we look at two polynomials

$$P(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$$

and

$$Q(X) = b_0 + b_1X + b_2X^2 + \cdots + b_mX^m,$$

then their product is a polynomial

$$c_0 + c_1X + c_2X^2 + \cdots + c_{n+m}X^{n+m},$$

where $c_0 = a_0b_0$, $c_1 = a_0b_1 + a_1b_0$, $c_2 = a_0b_2 + a_1b_1 + a_2b_0$, and in general

$$c_n = a_0b_n + a_1n_{n-1} + a_2b_{n-2} + \cdots + a_{n-1}b_1 + a_nb_0 = \sum_{k=0}^{n} a_kb_{n-k}.$$

- Can we use distributivity for an infinite sum?
- If we look at two polynomials

$$P(X) = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$$

and

$$Q(X) = b_0 + b_1 X + b_2 X^2 + \cdots + b_m X^m,$$

then their product is a polynomial

$$c_0 + c_1 X + c_2 X^2 + \cdots + c_{n+m} X^{n+m},$$

where $c_0 = a_0 b_0$, $c_1 = a_0 b_1 + a_1 b_0$, $c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$, and in general

$$c_n = a_0 b_n + a_1 n_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.$$

- This suggests the following definition.

- Definition 2. Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their Cauchy product is the series $\sum_{n=0}^{\infty} c_n$, where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$.

- Definition 2. Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their Cauchy product is the series $\sum_{n=0}^{\infty} c_n$, where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$.
- Remark: In spite of this intuitive idea, in general, the Cauchy product of two convergent series need not be convergent.

- Definition 2. Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their Cauchy product is the series $\sum_{n=0}^{\infty} c_n$, where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$.

- Remark: In spite of this intuitive idea, in general, the Cauchy product of two convergent series need not be convergent.

Example 5.

Consider the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, where

$$a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

- **Definition 2.** Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their **Cauchy product** is the series $\sum_{n=0}^{\infty} c_n$, where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$.

- **Remark:** In spite of this intuitive idea, in general, the Cauchy product of two convergent series need not be convergent.

  Example 5.
  Consider the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, where

$$a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

  Then $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent by the following result.
  (**Result:** The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, where $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of positive reals, is convergent if and only if $\lim_{n\to\infty} a_n = 0$.)

Claim: Their Cauchy product $\sum_{n=0}^{\infty} c_n$ is divergent.

Claim: Their Cauchy product $\sum_{n=0}^{\infty} c_n$ is divergent.
We have

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

Claim: Their Cauchy product $\sum_{n=0}^{\infty} c_n$ is divergent.
We have

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

For $0 \leq k \leq n$,

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2.$$

Claim: Their Cauchy product $\sum_{n=0}^{\infty} c_n$ is divergent.
We have

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

For $0 \le k \le n$,

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \le \left(\frac{n}{2}+1\right)^2.$$

$$\implies \quad \frac{1}{\sqrt{(k+1)(n-k+1)}} \ge \frac{2}{n+2}, \quad \text{for all } 0 \le k \le n$$

Claim: Their Cauchy product $\sum_{n=0}^{\infty} c_n$ is divergent.
We have

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

For $0 \leq k \leq n$,

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2.$$

$$\implies \quad \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \frac{2}{n+2}, \quad \text{for all } 0 \leq k \leq n$$

$$\implies \quad |c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \frac{2(n+1)}{n+2} = \frac{2(1+\frac{1}{n})}{1+\frac{2}{n}} \to 2.$$

Claim: Their Cauchy product $\sum_{n=0}^{\infty} c_n$ is divergent.
We have

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

For $0 \leq k \leq n$,

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2.$$

$$\implies \quad \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \frac{2}{n+2}, \quad \text{for all } 0 \leq k \leq n$$

$$\implies \quad |c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \frac{2(n+1)}{n+2} = \frac{2(1+\frac{1}{n})}{1+\frac{2}{n}} \to 2.$$

Therefore, it follows that $\sum_{n=0}^{\infty} c_n$ is not convergent.

Claim: Their Cauchy product $\sum_{n=0}^{\infty} c_n$ is divergent.
We have

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

For $0 \le k \le n$,

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \le \left(\frac{n}{2}+1\right)^2.$$

$$\implies \quad \frac{1}{\sqrt{(k+1)(n-k+1)}} \ge \frac{2}{n+2}, \quad \text{for all } 0 \le k \le n$$

$$\implies \quad |c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \ge \frac{2(n+1)}{n+2} = \frac{2(1+\frac{1}{n})}{1+\frac{2}{n}} \to 2.$$

Therefore, it follows that $\sum_{n=0}^{\infty} c_n$ is not convergent.

- However, things are not that bad. We will revisit this and see when can we assure that the Cauchy product of two series is convergent.

Tests for convergence of series

- n^{th} term test–already seen.

Tests for convergence of series

- n^{th} term test–already seen.
- Theorem 4. Let $\sum_{n=1}^{\infty} a_n$ be a series of non-negative real numbers. Then it is convergent if and only if its sequence of partial sums $\{s_n\}_{n \in \mathbb{N}}$ is bounded above. In this case

$$\sum_{n=1}^{\infty} a_n = \sup\{s_n : n \in \mathbb{N}\}.$$

Tests for convergence of series

- n^{th} term test–already seen.

- Theorem 4. Let $\sum_{n=1}^{\infty} a_n$ be a series of non-negative real numbers. Then it is convergent if and only if its sequence of partial sums $\{s_n\}_{n\in\mathbb{N}}$ is bounded above. In this case

$$\sum_{n=1}^{\infty} a_n = \sup\{s_n : n \in \mathbb{N}\}.$$

Proof: Exercise

- Theorem 5 (Comparison test). Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be real sequences, and suppose that there exists $N \in \mathbb{N}$ such that

$$0 \leq a_n \leq b_n, \ \forall n \geq N.$$

(i) If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.
(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$.

- **Theorem 5 (Comparison test).** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be real sequences, and suppose that there exists $N \in \mathbb{N}$ such that

$$0 \leq a_n \leq b_n, \ \forall n \geq N.$$

(i) If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.
(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$.

Proof: (i) Let $\epsilon > 0$ be arbitrary.

- **Theorem 5 (Comparison test).** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be real sequences, and suppose that there exists $N \in \mathbb{N}$ such that

$$0 \le a_n \le b_n, \ \forall n \ge N.$$

(i) If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.
(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$.

Proof: (i) Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} b_n$ is convergent, by Cauchy criterion, for the ϵ there exists $K \in \mathbb{N}$ such that

$$|b_{n+1} + b_{n+2} + \cdots + b_m| < \epsilon, \ \forall m > n \ge K.$$

- **Theorem 5 (Comparison test).** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be real sequences, and suppose that there exists $N \in \mathbb{N}$ such that

$$0 \leq a_n \leq b_n, \ \forall n \geq N.$$

(i) If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.
(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$.

Proof: (i) Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} b_n$ is convergent, by Cauchy criterion, for the ϵ there exists $K \in \mathbb{N}$ such that

$$|b_{n+1} + b_{n+2} + \cdots + b_m| < \epsilon, \ \forall m > n \geq K.$$

Then

$$0 \leq a_{n+1}+a_{n+2}+\cdots+a_m \leq b_{n+1}+b_{n+2}+\cdots+b_m < \epsilon, \ \forall m > n \geq M,$$

where $M := \max\{N, K\}$.

- **Theorem 5 (Comparison test).** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be real sequences, and suppose that there exists $N \in \mathbb{N}$ such that

$$0 \le a_n \le b_n, \ \forall n \ge N.$$

(i) If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.
(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$.

Proof: (i) Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} b_n$ is convergent, by Cauchy criterion, for the ϵ there exists $K \in \mathbb{N}$ such that

$$|b_{n+1} + b_{n+2} + \cdots + b_m| < \epsilon, \ \forall m > n \ge K.$$

Then

$$0 \le a_{n+1}+a_{n+2}+\cdots+a_m \le b_{n+1}+b_{n+2}+\cdots+b_m < \epsilon, \ \forall m > n \ge M,$$

where $M := \max\{N, K\}$. Since $\epsilon > 0$ is arbitrary, again by Cauchy criterion, it follows that $\sum_{n=1}^{\infty} a_n$ is convergent.

- **Theorem 5 (Comparison test).** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be real sequences, and suppose that there exists $N \in \mathbb{N}$ such that

$$0 \le a_n \le b_n, \ \forall n \ge N.$$

(i) If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.
(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$.

Proof: (i) Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} b_n$ is convergent, by Cauchy criterion, for the ϵ there exists $K \in \mathbb{N}$ such that

$$|b_{n+1} + b_{n+2} + \cdots + b_m| < \epsilon, \ \forall m > n \ge K.$$

Then

$$0 \le a_{n+1}+a_{n+2}+\cdots+a_m \le b_{n+1}+b_{n+2}+\cdots+b_m < \epsilon, \ \forall m > n \ge M,$$

where $M := \max\{N, K\}$. Since $\epsilon > 0$ is arbitrary, again by Cauchy criterion, it follows that $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) Follows from (i). □

- Example 6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

- Example 6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The series is convergent by result (i) of the comparison test. In fact, we have

$$0 \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2}, \quad \forall n \geq 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

- Example 6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The series is convergent by result (i) of the comparison test. In fact, we have

$$0 \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2}, \quad \forall n \geq 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Alternative proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

- Example 6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The series is convergent by result (i) of the comparison test. In fact, we have

$$0 \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2}, \quad \forall n \geq 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Alternative proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Then

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}, \ \forall n \in \mathbb{N}.$$

- Example 6. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$. The series is convergent by result (i) of the comparison test. In fact, we have

$$0 \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2}, \quad \forall n \geq 1$$

and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent.

Alternative proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$. Then

$$s_n = \sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1}, \ \forall n \in \mathbb{N}.$$

$$\Longrightarrow \lim_{n\to\infty} s_n = 1.$$

- Example 6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The series is convergent by result (i) of the comparison test. In fact, we have

$$0 \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2}, \quad \forall n \geq 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Alternative proof: Let $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Then

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1}, \; \forall n \in \mathbb{N}.$$

$$\Longrightarrow \lim_{n\to\infty} s_n = 1.$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

- Exercise: A series $\sum_{k=1}^{\infty} b_n$ is said to be a telescoping series if there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $b_n = a_{n+1} - a_n$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\lim_{n\to\infty} a_n$ exists. In such a case, find the sum.

- Exercise: A series $\sum_{k=1}^{\infty} b_n$ is said to be a telescoping series if there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $b_n = a_{n+1} - a_n$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\lim_{n\to\infty} a_n$ exists. In such a case, find the sum.

- Example 7. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n!}$.

- Exercise: A series $\sum_{k=1}^{\infty} b_n$ is said to be a telescoping series if there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $b_n = a_{n+1} - a_n$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\lim_{n\to\infty} a_n$ exists. In such a case, find the sum.

- Example 7. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n!}$.
  The series is convergent by result (i) of the comparison test. In fact, we have

$$0 \leq \frac{1}{n!} \leq \frac{1}{n^2}, \quad \forall n \geq 4$$

  and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

- Exercise: A series $\sum_{k=1}^{\infty} b_n$ is said to be a telescoping series if there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $b_n = a_{n+1} - a_n$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\lim\limits_{n\to\infty} a_n$ exists. In such a case, find the sum.

- Example 7. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n!}$.
The series is convergent by result (i) of the comparison test.
In fact, we have

$$0 \leq \frac{1}{n!} \leq \frac{1}{n^2}, \quad \forall n \geq 4$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Note: It can be proved that $\sum_{n=1}^{\infty} \frac{1}{n!} = e$ (Euler number).

- Exercise: A series $\sum_{k=1}^{\infty} b_n$ is said to be a telescoping series if there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $b_n = a_{n+1} - a_n$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\lim_{n\to\infty} a_n$ exists. In such a case, find the sum.

- Example 7. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n!}$.
  The series is convergent by result (i) of the comparison test.
  In fact, we have

$$0 \leq \frac{1}{n!} \leq \frac{1}{n^2}, \quad \forall n \geq 4$$

  and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

  Note: It can be proved that $\sum_{n=1}^{\infty} \frac{1}{n!} = e$ (Euler number).

- Exercise: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

- Theorem 6 (Limit comparison test): Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be strictly positive sequences.
  (i) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent.

- Theorem 6 (Limit comparison test): Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be strictly positive sequences.
  (i) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent.
  (ii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

- Theorem 6 (Limit comparison test): Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be strictly positive sequences.
(i) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
(iii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

- **Theorem 6 (Limit comparison test):** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be strictly positive sequences.
  (i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent.
  (ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  (iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

  Proof: (i) Since $c > 0$, there exists $K \in \mathbb{N}$ such that

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}, \ \forall n \geq K.$$

- **Theorem 6 (Limit comparison test):** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be strictly positive sequences.
  (i) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent.
  (ii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  (iii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

  Proof: (i) Since $c > 0$, there exists $K \in \mathbb{N}$ such that

$$\left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}, \ \forall n \geq K.$$

$$\implies \quad -\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}, \ \forall n \geq K.$$

- **Theorem 6 (Limit comparison test):** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be strictly positive sequences.
  (i) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent.
  (ii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  (iii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

  Proof: (i) Since $c > 0$, there exists $K \in \mathbb{N}$ such that

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}, \ \forall n \geq K.$$

$$\implies \quad -\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}, \ \forall n \geq K.$$

$$\implies \quad \left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n, \ \forall n \geq K.$$

- **Theorem 6 (Limit comparison test):** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be strictly positive sequences.
  (i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent.
  (ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  (iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

  Proof: (i) Since $c > 0$, there exists $K \in \mathbb{N}$ such that

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}, \ \forall n \geq K.$$

$$\implies \quad -\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}, \ \forall n \geq K.$$

$$\implies \quad \left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n, \ \forall n \geq K.$$

  Therefore, by comparison test, the result follows.

(ii) There exists $K \in \mathbb{N}$ such that

$$0 < \frac{a_n}{b_n} < 1, \ \forall n \geq K.$$

(ii) There exists $K \in \mathbb{N}$ such that

$$0 < \frac{a_n}{b_n} < 1, \ \forall n \geq K.$$

$$\Longrightarrow \quad 0 < a_n < b_n, \ \forall n \geq K.$$

(ii) There exists $K \in \mathbb{N}$ such that

$$0 < \frac{a_n}{b_n} < 1, \ \forall n \geq K.$$

$$\implies \quad 0 < a_n < b_n, \ \forall n \geq K.$$

Therefore, again by comparison test, the result follows.

(ii) There exists $K \in \mathbb{N}$ such that

$$0 < \frac{a_n}{b_n} < 1, \ \forall n \geq K.$$

$$\implies \quad 0 < a_n < b_n, \ \forall n \geq K.$$

Therefore, again by comparison test, the result follows.
(iii) Similar □

(ii) There exists $K \in \mathbb{N}$ such that

$$0 < \frac{a_n}{b_n} < 1, \ \forall n \geq K.$$

$$\implies \quad 0 < a_n < b_n, \ \forall n \geq K.$$

Therefore, again by comparison test, the result follows.
(iii) Similar □

Example 8. Test the convergence of the following series.
(i) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

(ii) There exists $K \in \mathbb{N}$ such that

$$0 < \frac{a_n}{b_n} < 1, \ \forall n \geq K.$$

$$\implies \quad 0 < a_n < b_n, \ \forall n \geq K.$$

Therefore, again by comparison test, the result follows.
(iii) Similar □

Example 8. Test the convergence of the following series.
(i) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Solution: (i) Let $a_n = \frac{2n+1}{n^2+2n+1}$ and $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$.

(ii) There exists $K \in \mathbb{N}$ such that

$$0 < \frac{a_n}{b_n} < 1, \ \forall n \geq K.$$

$$\implies \quad 0 < a_n < b_n, \ \forall n \geq K.$$

Therefore, again by comparison test, the result follows.
(iii) Similar □

Example 8. Test the convergence of the following series.
(i) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Solution: (i) Let $a_n = \frac{2n+1}{n^2+2n+1}$ and $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2.$$

(ii) There exists $K \in \mathbb{N}$ such that

$$0 < \frac{a_n}{b_n} < 1, \ \forall n \geq K.$$

$$\implies \quad 0 < a_n < b_n, \ \forall n \geq K.$$

Therefore, again by comparison test, the result follows.
(iii) Similar □

Example 8. Test the convergence of the following series.
(i) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Solution: (i) Let $a_n = \frac{2n+1}{n^2+2n+1}$ and $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2.$$

Since $\sum_{n=1}^{\infty} b_n$ is divergent, by result (i) of Limit comparison test, it follows that $\sum_{n=1}^{\infty} a_n$ is divergent.

(ii) There exists $K \in \mathbb{N}$ such that

$$0 < \frac{a_n}{b_n} < 1, \; \forall n \geq K.$$

$$\implies \quad 0 < a_n < b_n, \; \forall n \geq K.$$

Therefore, again by comparison test, the result follows.
(iii) Similar □

Example 8. Test the convergence of the following series.
(i) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Solution: (i) Let $a_n = \frac{2n+1}{n^2+2n+1}$ and $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2.$$

Since $\sum_{n=1}^{\infty} b_n$ is divergent, by result (i) of Limit comparison test, it follows that $\sum_{n=1}^{\infty} a_n$ is divergent.(ii) Exercise. (Hint: Compare with $\{\frac{1}{2^n}\}_{n\in\mathbb{N}}$).