ANALYSIS-I

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Recall

- Definition. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

  An expression of the form $\sum_{n=1}^{\infty} a_n$ is called an infinite series.

  For each $n \in \mathbb{N}$, the finite sum $s_n = \sum_{k=1}^{n} a_k$ is called the n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$.

  The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if $\{s_n\}_{n\in\mathbb{N}}$ is convergent.

  In such a case, the limit $s := \lim_{n\to\infty} s_n$ is called the sum of the series, and we denote this fact by the symbol $\sum_{n=1}^{\infty} a_n = s$.

  The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be divergent if $\{s_n\}_{n\in\mathbb{N}}$ is divergent.

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  The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be divergent if $\{s_n\}_{n\in\mathbb{N}}$ is divergent.
- Theorem (Cauchy criterion). An infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if for every $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that $|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon, \quad \forall m > n \geq K$.

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- Theorem (Comparison test). Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be real sequences, and suppose that there exists $N \in \mathbb{N}$ such that

$$0 \leq a_n \leq b_n, \ \forall n \geq N.$$

  (i) If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.
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- Theorem 6 (Limit comparison test): Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be strictly positive sequences.
  (i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum_{n=1}^{\infty} b_n$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent.
  (ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  (iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Absolute and Conditional convergence

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  (iii) $\sum_{n=1}^{\infty} (-1)^{n+1}$ is neither absolutely convergent nor conditionally convergent.

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  Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, by Cauchy criterion, there exists $K \in \mathbb{N}$ such that

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Then for all $m > n \leq K$, we have

$$|a_{n+1} + a_{n+2} + \cdots + a_m| \quad \leq \quad |a_{n+1}| + |a_{n+2}| + \cdots + |a_m|$$

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Since $\epsilon > 0$ is arbitrary, Cauchy criterion implies that $\sum_{n=1}^{\infty} a_n$ is convergent. ■

Tests for absolute convergence

- Theorem (Cauchy's Root Test). Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence.
  (i) If there exist $r \in \mathbb{R}$ with $r < 1$ and $K \in \mathbb{N}$ such that
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  Proof: (i) Since (1) holds, we have $|a_n| \leq r^n, \ \forall n \geq K$. Now, since $r < 1$, the geometric series $\sum_{n=1}^{\infty} r^n$ is convergent. Therefore, by comparison test, the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

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  This implies that $a_n \nrightarrow 0$ as $n \to \infty$.
  Therefore, by n^{th} term test, the series $\sum_{n=1}^{\infty} a_n$ is divergent. ■

- **Corollary (Cauchy's Root Test–another version).**
  Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence and suppose that

$$r := \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \tag{3}$$

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exists in $\mathbb{R}$.

(i) If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
(ii) If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof: (i) Since $r < 1$, we can choose $s \in \mathbb{R}$ such that
$r < s < 1$.
Since (3) holds, there exists $K \in \mathbb{N}$ such that

$$\left| |a_n|^{\frac{1}{n}} - r \right| < s - r, \ \forall n \geq K.$$

$\implies |a_n|^{\frac{1}{n}} - r < s - r, \ \forall n \geq K.$
$\implies |a_n|^{\frac{1}{n}} < s, \ \forall n \geq K.$
Since $s < 1$, by (i) of the previous theorem, it follows that
$\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

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$\implies s < |a_n|^{\frac{1}{n}}, \ \forall n \geq K.$
Since $s > 1$, by (ii) of the previous theorem, we get that $\sum_{n=1}^{\infty} a_n$ is divergent. ■

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(i) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (ii) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^{2021}}$

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- **Theorem (D'Alembert Ratio Test).** Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonzero real numbers.
  (i) If there exist $r \in \mathbb{R}$ with $0 < r < 1$ and $K \in \mathbb{N}$ such that

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Now, since $r < 1$, the geometric series $\sum_{n=1}^{\infty} r^n$ is convergent. This implies that the series $\sum_{n=1}^{\infty} \frac{|a_K|}{r^K} r^n$ is convergent. Therefore, by comparison test, $\sum_{n=1}^{\infty} |a_n|$ is convergent.

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This implies that $a_n \nrightarrow 0$ as $n \to \infty$, since $|a_K| > 0$.
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  Proof: Exercise.

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$$\frac{\frac{2^{n+1}+7}{5^{n+1}}}{\frac{2^n+7}{5^n}} = \frac{1}{5} \cdot \frac{2^{n+1}+7}{2^n+7} = \frac{1}{5} \cdot \frac{2+\frac{7}{2^n}}{1+\frac{7}{2^n}} \to \frac{1}{5} \cdot \frac{2}{1} = \frac{2}{5} < 1.$$

  (ii) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ is divergent by ratio test, because

$$\frac{\frac{(2n+2)!}{(n+1)!(n+1)!}}{\frac{(2n)!}{n!n!}} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} = \frac{4+\frac{2}{n}}{1+\frac{1}{n}} \to 4 > 1.$$

- Remark: The test is inconclusive if $r = 1$.
  For $\sum_{n=1}^{\infty} \frac{1}{n}$: $\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \to 1$
  For $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \left(\frac{n}{n+1}\right)^2 \to 1^2 = 1$

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- The answer is NO, as seen from the next result.

Convergence of Cauchy product

- **Theorem (Mertens' Theorem).** Let $\sum_{n=0}^{\infty} a_n$ be absolutely convergent and $\sum_{n=0}^{\infty} b_n$ be convergent. If $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = b$, then their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent and $\sum_{n=0}^{\infty} c_n = ab$.

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  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$, $\{t_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, and $\sum_{n=0}^{\infty} c_n$, respectively.

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$$\begin{aligned}
u_n &= c_0 + c_1 + \cdots + c_n \\
&= (a_0 b_0) + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\
&= a_0(b_0 + \cdots + b_n) + a_1(b_0 + \cdots + b_{n-1}) + \cdots + a_n b_0 \\
&= a_0 t_n + a_1 t_{n-1} + \cdots + a_n t_0 \\
&= a_0 t_n + a_1 t_{n-1} + \cdots + a_n t_0 - \left(\sum_{k=0}^{n} a_k\right) b + s_n b \\
&= a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b) + s_n b,
\end{aligned}$$

i.e.,

$$\begin{aligned} c_n &= a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b) + s_n b \\ &= v_n + s_n b, \end{aligned} \tag{7}$$

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$$|t_n - b| < \epsilon, \ \forall n \geq K_1.$$

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Since $\{t_n - b\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded, there exists $M > 0$ such that

$$|t_n - b| \leq M, \ \forall n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, say $\sum_{n=1}^{\infty} |a_n| = \alpha$,by Cauchy criterion there exists $K_2 \in \mathbb{N}$ such that

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&\quad + |a_{n-K+1}||t_{n+K-1} - b| + \cdots + |a_n||t_0 - b| \\
&\leq (|a_0| + |a_1| + \cdots + |a_{n-K}|)\epsilon \\
&\quad + (|a_{n-K+1}| + \cdots + |a_n|)M \\
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Since $\epsilon > 0$ is arbitrary, it follows that $\lim_{n\to\infty} v_n = 0$. This completes the proof.