ANALYSIS-I

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Recall

- **Definition.** Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that $\sum_{n=1}^{\infty} a_n$ is
  - (i) absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent;
  - (ii) conditionally convergent if it is convergent, but not absolutely convergent.

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- Theorem. Every absolutely convergent series is convergent.
- Theorem (Cauchy's Root Test).
  Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence and suppose that

$$r := \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

  exists in $\mathbb{R}$.
  (i) If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
  (ii) If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- **Theorem (D'Alembert Ratio Test).**
  Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonzero real numbers and suppose that

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- **Definition.** Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their **Cauchy product** is the series $\sum_{n=0}^{\infty} c_n$, where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$.

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- Remark. The Cauchy product of two convergent series need not be convergent.

Convergence of Cauchy product

- **Theorem (Mertens' Theorem).** Let $\sum_{n=0}^{\infty} a_n$ be absolutely convergent and $\sum_{n=0}^{\infty} b_n$ be convergent. If $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = b$, then their Cauchy product $\sum_{n=0}^{\infty} c_n$ is convergent and $\sum_{n=0}^{\infty} c_n = ab$.

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  Proof: Let $\{s_n\}_{n\in\mathbb{N}}$, $\{t_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, and $\sum_{n=0}^{\infty} c_n$, respectively.

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$$\begin{aligned}
u_n &= c_0 + c_1 + \cdots + c_n \\
&= (a_0 b_0) + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\
&= a_0(b_0 + \cdots + b_n) + a_1(b_0 + \cdots + b_{n-1}) + \cdots + a_n b_0 \\
&= a_0 t_n + a_1 t_{n-1} + \cdots + a_n t_0 \\
&= a_0 t_n + a_1 t_{n-1} + \cdots + a_n t_0 - \left(\sum_{k=0}^{n} a_k\right) b + s_n b \\
&= a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b) + s_n b,
\end{aligned}$$

i.e.,

$$\begin{aligned} c_n &= a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b) + s_n b \\ &= v_n + s_n b, \end{aligned} \tag{1}$$

where $v_n = a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b)$ for all $n \in \mathbb{N} \cup \{0\}$.

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Since $\{t_n - b\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded, there exists $M > 0$ such that

$$|t_n - b| \leq M, \ \forall n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, say $\sum_{n=1}^{\infty} |a_n| = \alpha$,by Cauchy criterion there exists $K_2 \in \mathbb{N}$ such that

$$|a_{n+1}| + |a_{n+2}| + \cdots |a_m| < \epsilon, \ \forall m > n \geq K_2.$$

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Let $K := \max\{K_1, K_2\}$. Then for all $n \geq 2K$, we have

$$\begin{aligned}
|v_n| &= |a_0(t_n - b) + a_1(t_{n-1} - b) + \cdots + a_n(t_0 - b)| \\
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&\quad + |a_{n-K+1}||t_{n+K-1} - b| + \cdots + |a_n||t_0 - b| \\
&\leq (|a_0| + |a_1| + \cdots + |a_{n-K}|)\epsilon \\
&\quad + (|a_{n-K+1}| + \cdots + |a_n|)M \\
&\leq \alpha\epsilon + \epsilon M \\
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Since $\epsilon > 0$ is arbitrary, it follows that $\lim_{n \to \infty} v_n = 0$. This completes the proof.

Tests for conditional convergence

- Definition. A sequence $\{a_n\}_{n\in\mathbb{N}}$ of non-negative real numbers is said to be alternating if $(-1)^{n+1}a_n$ is non-negative for all $n \in \mathbb{N}$.

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- Theorem (Alternating Series Test). Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive reals such that $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

- Theorem (Dirichlet's Test). Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of reals with $\lim_{n\to\infty} a_n = 0$ and let the sequence of partial sums $\{s_n\}_{n\in\mathbb{N}}$ of $\sum_{n=1}^{\infty} b_n$ be bounded. Then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

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  Abel's Lemma. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of reals and $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} b_n$ with $s_0 := 0$. If $m > n$, then

$$\sum_{k=n+1}^{m} a_k b_k = (a_m s_m - a_{n+1} s_n) + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) s_k. \qquad (2)$$

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  Proof of the lemma:

$$\sum_{k=n+1}^{m} a_k b_k = \sum_{k=n+1}^{m} a_k (s_k - s_{k-1})$$

$$= -a_{n+1} s_n + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) s_k + a_m s_m = \text{RHS of (2)}$$

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$$\begin{aligned}\left|\sum_{k=n+1}^{m} a_k b_k\right| &= \left|(a_m s_m - a_{n+1} s_n) + \sum_{k=n+1}^{m-1}(a_k - a_{k+1})s_k\right| \\ &\leq |a_m||s_m| + |a_{n+1}||s_n| + \sum_{k=n+1}^{m-1}|a_k - a_{k+1}||s_k| \\ &\leq (a_m + a_{n+1})M + \sum_{k=n+1}^{m-1}(a_k - a_{k+1})M \\ &= \{(a_m + a_{n+1}) + (a_{n+1} - a_m)\}M = 2a_{n+1}M \qquad (3)\end{aligned}$$

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Therefore, by (3) we have $|\sum_{k=n+1}^{m} a_k b_k| < \epsilon$, $\forall m > n \geq K$.
Since $\epsilon > 0$ is arbitrary, by Cauchy criterion, it follows that $\sum_{n=1}^{\infty} a_n b_n$ is convergent. ■

Theorem (Abel's Test). Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent monotone sequence and let the series $\sum_{n=1}^{\infty} b_n$ be convergent. Then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

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Therefore, by Dirichlet's test, the series $\sum_{n=1}^{\infty} u_n b_n$ is convergent.

Theorem (Abel's Test). Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent monotone sequence and let the series $\sum_{n=1}^{\infty} b_n$ be convergent. Then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Proof:

Case (i): Let $\{a_n\}_{n\in\mathbb{N}}$ be decreasing with limit a.
Set $u_n = a_n - a, \forall n \in \mathbb{N}$.
Then

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Now, $\{u_n\}_{n\in\mathbb{N}}$ is decreasing with limit 0 and the sequence of partial sums of $\sum_{n=1}^{\infty} b_n$. is bounded.
Therefore, by Dirichlet's test, the series $\sum_{n=1}^{\infty} u_n b_n$ is convergent.
This implies by (4) that the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent, because by hypothesis $\sum_{n=1}^{\infty} b_n$ is convergent.

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(i) $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$ is convergent by Dirichlet's test.

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$ is convergent by Abel's test.

Grouping of series

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  Proof: Exercise

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$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} + \cdots$$

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- Let s_n be the n^{th} partial sum of the original series and t_n be the n^{th} partial sum of this rearranged series.
- Then

$$\begin{aligned} t_{3n} &= \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}\right) + \cdots \\ &= \left(\frac{1}{2} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{4n-2} - \frac{1}{4n}\right) + \cdots \\ &= \frac{s_{2n}}{2} \to \frac{s}{2} \end{aligned}$$

- $$t_{3n+1} = t_{3n} + \frac{1}{2n+1} = \frac{s_{2n}}{2} + \frac{1}{2n+1} \to \frac{s}{2}$$
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- Thus the rearranged series may converge to a sum different from that of the given series.
- Definition. A series $\sum_{n=1}^{\infty} b_n$ is said to be a rearrangement of a series $\sum_{n=1}^{\infty} a_n$ if there is a bijection f of $\mathbb{N}$ onto $\mathbb{N}$ such that $b_k = a_{f(k)}$ for all $k \in \mathbb{N}$.