ANALYSIS-I

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Rearrangements of series

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- Thus the rearranged series may converge to a sum different from that of the given series.
- However, things are not that bad when we deal with absolutely convergent series.

- **Theorem (Rearrangement theorem).** If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then any rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ converges to the same value.

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Since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{n \to \infty} t_n = a$. $\blacksquare$.

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- This theorem should convince us of the danger of manipulating an infinite series without any attention to rigorous analysis.
- To prove this theorem, we need the notions of positive and negative parts of a series.

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- Note that all the terms of both these series are non-negative.
- For example, if $a_n = \frac{(-1)^{n+1}}{n}$, then

$$\sum_{n=1}^{\infty} a_n^+ = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots$$

and

$$\sum_{n=1}^{\infty} a_n^- = 0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + \cdots$$

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  Therefore we have

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  By hypothesis $\sum_{n=1}^{\infty} |a_n|$ is divergent, which implies that $\lim_{n\to\infty} t_n = \infty$.

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Since $\lim_{n\to\infty} t_n = \infty$, there exists $K \in \mathbb{N}$ such that

$$t_n > 2r + M, \ \forall n \geq K.$$

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Then $t_n + s_n > (2r + M) + (-M) = 2r, \forall n \geq K$, implying that

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Therefore $\lim_{n\to\infty} u_n^+ = \infty$.

Note that

$$u_n^+ = \frac{t_n + s_n}{2} \quad \text{and} \quad u_n^- = \frac{t_n - s_n}{2} \quad \text{for all } n \in \mathbb{N}$$

We shall show that $\lim_{n\to\infty} u_n^+ = \infty$.
Let $r > 0$ be arbitrary.
Since $\{s_n\}_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that

$$-M \le s_n \le M, \ \forall n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} t_n = \infty$, there exists $K \in \mathbb{N}$ such that

$$t_n > 2r + M, \ \forall n \ge K.$$

Then $t_n + s_n > (2r + M) + (-M) = 2r, \forall n \ge K$, implying that

$$u_n^+ = \frac{t_n + s_n}{2} > \frac{2r}{2} = r, \ \forall n \ge K.$$

Therefore $\lim_{n\to\infty} u_n^+ = \infty$. A similar argument shows that $\lim_{n\to\infty} u_n^- = \infty$. ■

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- Reference: Theorem 3.54 in [Walter Rudin, Principles of Mathematical Analysis, Third Edition, McGraw Hill Inc., 1976]

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  Theorem 8.33 in [Tom M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, Inc., 1974]

Infinite products

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- By analogy with infinite series, it seems natural to call the product $\prod_{n=1}^{\infty} a_n$ converges if $\{p_n\}_{n\in\mathbb{N}}$ converges.
- However, this definition is inconvenient since every product having one factor zero would converge regardless of the behavior of the other factors.

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  (iii) If there exists an $N \in \mathbb{N}$ such that $n > N$ implies $a_n \neq 0$, then we say that $\prod_{n=1}^{\infty} a_n$ is convergent provided that $\prod_{n=N+1}^{\infty} a_n$ converges as described in (ii).

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  In this case the value of the product $\prod_{n=1}^{\infty} a_n$ is

$$a_1 a_2 \cdots a_N \prod_{n=N+1}^{\infty} a_n.$$

  (iv) $\prod_{n=1}^{\infty} a_n$ is called divergent if it does not converge as described in (ii) or (iii).

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- Theorem (Cauchy criterion). The infinite product $\prod_{n=1}^{\infty} a_n$ is convergent if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

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- Theorem. If $\prod_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 1$.
- For this reason, the factors of a product are written as $1 + a_n$ instead of just a_n. Thus, if $\prod_{n=1}^{\infty}(1 + a_n)$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

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